CHAPTER 10
Analytic Geometry in Three Dimensions

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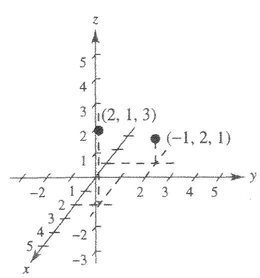
Section 10.1 The Three-Dimensional Coordinate System

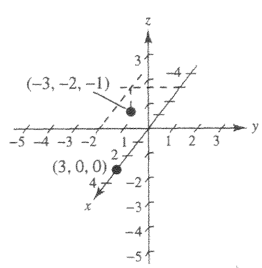
- You should be able to plot points in the three-dimensional coordinate system.
- The distance between the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is
  $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.
- The midpoint of the line segment joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is
  $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$.
- The equation of the sphere with center (h, k, j) and radius r is
  $(x - h)^2 + (y - k)^2 + (z - j)^2 = r^2$.
- You should be able to find the trace of a surface in space.

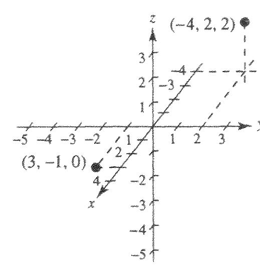
Vocabulary Check

1. three-dimensional
2. xy-plane, xz-plane, yz-plane
3. octants
4. Distance Formula
5. $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$
6. sphere
7. surface, space
8. trace

1. $A(-1, 4, 3), B(1, 3, -2), C(-3, 0, -2)$
2. $A(6, 2, -3), B(2, -1, 2)\ C(-2, 3, 0)$
3. $A(-2, -1, 4), B(3, -2, 0), C(-2, 2, -3)$
4. $A(0, 5, -3), B(5, -4, -2), C(-4, 1, 5)$

5. 

6. 

7. 

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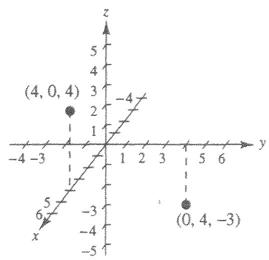

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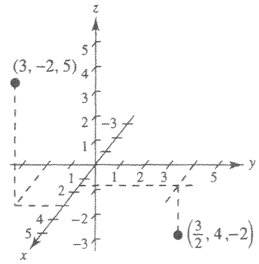

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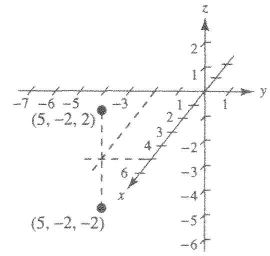


11. $x = -3, y = 3, z = 4$: $(-3, 3, 4)$

12. $x = 6, y = -1, z = -1 \implies (6, -1, -1)$

13. $y = z = 0, x = 10$: $(10, 0, 0)$

14. $x = 0, y = 2, z = 8 \implies (0, 2, 8)$

15. Octant IV

16. Octant VI

17. Octants I, II, III, IV (above the xy-plane)

18. Octants III, IV, VII, or VIII

19. Octants II, IV, VI, VIII

20. Octants I, II, VII, or VIII

21.
$$\begin{aligned} d &= \sqrt{(7-3)^2 + (4-2)^2 + (8-(-5))^2} \\ &= \sqrt{4^2 + 2^2 + 13^2} \\ &= \sqrt{16 + 4 + 169} \\ &= \sqrt{189} \\ &= 3\sqrt{21} \approx 13.748 \end{aligned}$$

22.
$$\begin{aligned} d &= \sqrt{(4-2)^2 + (1-1)^2 + (9-6)^2} \\ &= \sqrt{4+9} \\ &= \sqrt{13} \end{aligned}$$

23.
$$\begin{aligned} d &= \sqrt{[6-(-1)]^2 + [0-4]^2 + [-9-(-2)]^2} \\ &= \sqrt{7^2 + 4^2 + 7^2} \\ &= \sqrt{49 + 16 + 49} \\ &= \sqrt{114} \\ &\approx 10.677 \end{aligned}$$

24.
$$\begin{aligned} d &= \sqrt{(1-(-2))^2 + (1-(-3))^2 + (-7-(-7))^2} \\ &= \sqrt{9+16} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

25.
$$\begin{aligned} d &= \sqrt{(1-0)^2 + [0-(-3)]^2 + (-10-0)^2} \\ &= \sqrt{1 + 9 + 100} \\ &= \sqrt{110} \approx 10.488 \end{aligned}$$

26.
$$\begin{aligned} d &= \sqrt{(2-0)^2 + (-4-6)^2 + (0-(-3))^2} \\ &= \sqrt{4 + 100 + 9} \\ &= \sqrt{113} \end{aligned}$$

27.
$$\begin{aligned} d_1 &= \sqrt{(0-0)^2 + (0-4)^2 + (2-0)^2} = \sqrt{20} = 2\sqrt{5} \\ d_2 &= \sqrt{(0-(-2))^2 + (0-5)^2 + (2-2)^2} = \sqrt{29} \\ d_3 &= \sqrt{(-2-0)^2 + (5-4)^2 + (2-0)^2} = 3 \\ d_1^2 + d_3^2 &= 20 + 9 = 29 = d_2^2 \end{aligned}$$

28.
$$\begin{aligned} d_1 &= \sqrt{(2-(-2))^2 + (-1-5)^2 + (2-0)^2} = \sqrt{56} = 2\sqrt{14} \\ d_2 &= (2-(-4))^2 + (-1-4)^2 + (2-1)^2 = \sqrt{62} \\ d_3 &= \sqrt{(-4-(-2))^2 + (4-5)^2 + (1-0)^2} = \sqrt{6} \\ d_1^2 + d_3^2 &= 56 + 6 = 62 = d_2^2 \end{aligned}$$

29. $d_1 = \sqrt{(2-0)^2 + (2-0)^2 + (1-0)^2} = \sqrt{9} = 3$

$d_2 = \sqrt{(2-2)^2 + (-4-2)^2 + (4-1)^2} = \sqrt{45} = 3\sqrt{5}$

$d_3 = \sqrt{(2-0)^2 + (-4-0)^2 + (4-0)^2} = \sqrt{36} = 6$

$d_1{}^2 + d_3{}^2 = 9 + 36 = 45 = d_2{}^2$

30. $d_1 = \sqrt{(1-1)^2 + (3-0)^2 + (1-1)^2} = \sqrt{9} = 3$

$d_2 = \sqrt{(1-1)^2 + (3-0)^2 + (1-3)^2} = \sqrt{13}$

$d_3 = \sqrt{(1-1)^2 + (0-0)^2 + (3-1)^2} = 2$

$d_1{}^2 + d_3{}^2 = 9 + 4 = 13 = d_2{}^2$

31. $d_1 = \sqrt{(5-1)^2 + (-1+3)^2 + (2+2)^2} = \sqrt{16+4+16} = \sqrt{36} = 6$

$d_2 = \sqrt{(5+1)^2 + (-1-1)^2 + (2-2)^2} = \sqrt{36+4} = \sqrt{40} = 2\sqrt{10}$

$d_3 = \sqrt{(-1-1)^2 + (1+3)^2 + (2+2)^2} = \sqrt{4+16+16} = \sqrt{36} = 6$

$d_1 = d_3$, Isosceles triangle

32. $d_1 = \sqrt{(7-5)^2 + (1-3)^2 + (3-4)^2} = \sqrt{4+4+1} = \sqrt{9} = 3$

$d_2 = \sqrt{(3-7)^2 + (5-1)^2 + (3-3)^2} = \sqrt{16+16} = \sqrt{32} = 4\sqrt{2}$

$d_3 = \sqrt{(3-5)^2 + (5-3)^2 + (3-4)^2} = \sqrt{4+4+1} = \sqrt{9} = 3$

$d_1 = d_3 = 3$, Isosceles triangle

33. $d_1 = \sqrt{(8-4)^2 + (1+1)^2 + (2+2)^2} = \sqrt{36} = 6$

$d_2 = \sqrt{(8-2)^2 + (1-3)^2 + (2-2)^2} = \sqrt{40} = 2\sqrt{10}$

$d_3 = \sqrt{(4-2)^2 + (-1-3)^2 + (-2-2)^2} = \sqrt{36} = 6$

Since $d_1 = d_3$, the triangle is isosceles.

34. $d_1 = \sqrt{(3-1)^2 + (0+2)^2 + (0+1)^2} = \sqrt{9} = 3$

$d_2 = \sqrt{(3-3)^2 + (0+6)^2 + (0-3)^2} = \sqrt{45} = 3\sqrt{5}$

$d_3 = \sqrt{(3-1)^2 + (-6+2)^2 + (3+1)^2} = \sqrt{36} = 6$

$d_1{}^2 + d_3{}^2 = 9 + 36 = 45 = d_2{}^2$

Right triangle

35. Midpoint: $\left(\frac{3-3}{2}, \frac{-6+4}{2}, \frac{10+4}{2}\right) = (0, -1, 7)$

36. Midpoint: $\left(\frac{-1+3}{2}, \frac{5+7}{2}, \frac{-3-1}{2}\right) = (1, 6, -2)$

37. Midpoint: $\left(\frac{6-4}{2}, \frac{-2+2}{2}, \frac{5+6}{2}\right) = \left(1, 0, \frac{11}{2}\right)$

38. Midpoint: $\left(\frac{-3-6}{2}, \frac{5+4}{2}, \frac{5+8}{2}\right) = \left(-\frac{9}{2}, \frac{9}{2}, \frac{13}{2}\right)$

39. Midpoint: $\left(\frac{-2+7}{2}, \frac{8-4}{2}, \frac{10+2}{2}\right) = \left(\frac{5}{2}, 2, 6\right)$

40. Midpoint: $\left(\frac{9+9}{2}, \frac{-5-2}{2}, \frac{1-4}{2}\right) = \left(9, -\frac{7}{2}, -\frac{3}{2}\right)$

41. $(x-3)^2 + (y-2)^2 + (z-4)^2 = 16$

42. $(x+3)^2 + (y-4)^2 + (z-3)^2 = 4$

43. $(x + 1)^2 + (y - 2)^2 + z^2 = 3$

44. $x^2 + (y + 1)^2 + (z - 3)^2 = 5$

45. $(x - 0)^2 + (y - 4)^2 + (z - 3)^2 = 3^2$

$x^2 + (y - 4)^2 + (z - 3)^2 = 9$

46. $(x - 2)^2 + (y + 1)^2 + (z - 8)^2 = 36$

47. Radius $= \dfrac{\text{Diameter}}{2} = 5$

$(x + 3)^2 + (y - 7)^2 + (z - 5)^2 = 5^2 = 25$

48. Radius $= \dfrac{\text{Diameter}}{2} = 4$

$(x - 0)^2 + (y - 5)^2 + (z + 9)^2 = 4^2 = 16$

49. Center: $\left(\dfrac{3+0}{2}, \dfrac{0+0}{2}, \dfrac{0+6}{2}\right) = \left(\dfrac{3}{2}, 0, 3\right)$

Radius: $\sqrt{\left(3 - \dfrac{3}{2}\right)^2 + (0 - 0)^2 + (0 - 3)^2} = \sqrt{\dfrac{9}{4} + 9} = \sqrt{\dfrac{45}{4}}$

Sphere: $\left(x - \dfrac{3}{2}\right)^2 + (y - 0)^2 + (z - 3)^2 = \dfrac{45}{4}$

50. Center: $\left(\dfrac{2-1}{2}, \dfrac{-2+4}{2}, \dfrac{2+6}{2}\right) = \left(\dfrac{1}{2}, 1, 4\right)$

Radius: $\sqrt{\left(2 - \dfrac{1}{2}\right)^2 + (-2 - 1)^2 + (2 - 4)^2} = \sqrt{\dfrac{9}{4} + 9 + 4} = \sqrt{\dfrac{61}{4}}$

Sphere: $\left(x - \dfrac{1}{2}\right)^2 + (y - 1)^2 + (z - 4)^2 = \dfrac{61}{4}$

51. $\left(x^2 - 5x + \frac{25}{4}\right) + y^2 + z^2 = \frac{25}{4}$

$\left(x - \frac{5}{2}\right)^2 + y^2 + z^2 = \frac{25}{4}$

Center: $\left(\frac{5}{2}, 0, 0\right)$

Radius: $\frac{5}{2}$

52. $x^2 + y^2 - 8y + 16 + z^2 = 16$

$x^2 + (y - 4)^2 + z^2 = 16$

Center: $(0, 4, 0)$

Radius: 4

53. $(x^2 - 4x + 4) + (y^2 + 2y + 1) + z^2 = 4 + 1$

$(x - 2)^2 + (y + 1)^2 + z^2 = 5$

Center: $(2, -1, 0)$

Radius: $\sqrt{5}$

54. $\left(x^2 - x + \dfrac{1}{4}\right) + \left(y^2 - y + \dfrac{1}{4}\right) + \left(z^2 - z + \dfrac{1}{4}\right) = \dfrac{1}{4} + \dfrac{1}{4} + \dfrac{1}{4}$

$\left(x - \dfrac{1}{2}\right)^2 + \left(y - \dfrac{1}{2}\right)^2 + \left(z - \dfrac{1}{2}\right)^2 = \dfrac{3}{4}$

Center: $\left(\dfrac{1}{2}, \dfrac{1}{2}, \dfrac{1}{2}\right)$

Radius: $\dfrac{\sqrt{3}}{2}$

55. $(x^2 - 4x + 4) + (y^2 + 2y + 1) + (z^2 - 6z + 9) = -10 + 4 + 1 + 9$

$$(x - 2)^2 + (y + 1)^2 + (z - 3)^2 = 4$$

Center: $(2, -1, 3)$

Radius: 2

56. $(x^2 - 6x + 9) + (y^2 + 4y + 4) + z^2 = -9 + 9 + 4$

$$(x - 3)^2 + (y + 2)^2 + z^2 = 4$$

Center: $(3, -2, 0)$

Radius: 2

57. $(x^2 + 4x + 4) + y^2 + (z^2 - 8z + 16) = -19 + 4 + 16$

$$(x + 2)^2 + y^2 + (z - 4)^2 = 1$$

Center: $(-2, 0, 4)$

Radius: 1

58. $x^2 + (y^2 - 8y + 16) + (z^2 - 6z + 9) = -13 + 16 + 9$

$$x^2 + (y - 4)^2 + (z - 3)^2 = 12$$

Center: $(0, 4, 3)$

Radius: $\sqrt{12} = 2\sqrt{3}$

59.
$$x^2 + y^2 + z^2 - 2x - \tfrac{2}{3}y - 8z = -\tfrac{73}{9}$$
$$(x^2 - 2x + 1) + \left(y^2 - \tfrac{2}{3}y + \tfrac{1}{9}\right) + (z^2 - 8z + 16) = -\tfrac{73}{9} + 1 + \tfrac{1}{9} + 16$$
$$(x - 1)^2 + \left(y - \tfrac{1}{3}\right)^2 + (z - 4)^2 = 9$$

Center: $\left(1, \frac{1}{3}, 4\right)$

Radius: 3

60.
$$x^2 + y^2 + z^2 - x - 3y - 2z = -\tfrac{5}{2}$$
$$\left(x^2 - x + \tfrac{1}{4}\right) + \left(y^2 - 3y + \tfrac{9}{4}\right) + (z^2 - 2z + 1) = -\tfrac{5}{2} + \tfrac{1}{4} + \tfrac{9}{4} + 1$$
$$\left(x - \tfrac{1}{2}\right)^2 + \left(y - \tfrac{3}{2}\right)^2 + (z - 1)^2 = 1$$

Center: $\left(\frac{1}{2}, \frac{3}{2}, 1\right)$

Radius: 1

61. $4(x^2 - 2x + 1) + 4(y^2 + 4y + 4) + 4z^2 = 4 + 16 + 1$

$$(x - 1)^2 + (y + 2)^2 + z^2 = \frac{21}{4}$$

Center: $(1, -2, 0)$

Radius: $\dfrac{\sqrt{21}}{2}$

62. $$9(x^2 - 2x + 1) + 9(y^2 + 4y + 4) + 9(z^2 + 6z + 9) = 9 + 36 + 81 + 126$$
$$9(x-1)^2 + 9(y+2)^2 + 9(z+3)^2 = 252$$
$$(x-1)^2 + (y+2)^2 + (z+3)^2 = 28 = \left(2\sqrt{7}\right)^2$$

Center: $(1, -2, -3)$

Radius: $2\sqrt{7}$

63. $$9x^2 - 6x + 9y^2 + 18y + 9z^2 = -1$$
$$x^2 - \tfrac{2}{3}x + \tfrac{1}{9} + y^2 + 2y + 1 + z^2 = -\tfrac{1}{9} + \tfrac{1}{9} + 1$$
$$\left(x - \tfrac{1}{3}\right)^2 + (y+1)^2 + z^2 = 1$$

Center: $\left(\frac{1}{3}, -1, 0\right)$

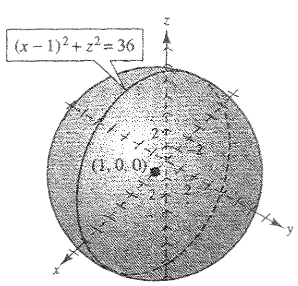
Radius: 1

64. $$x^2 - x + \frac{1}{4} + y^2 - 8y + 16 + z^2 + 2z + 1 = \frac{-33}{4} + \frac{1}{4} + 16 + 1$$
$$\left(x - \frac{1}{2}\right)^2 + (y-4)^2 + (z+1)^2 = 9$$

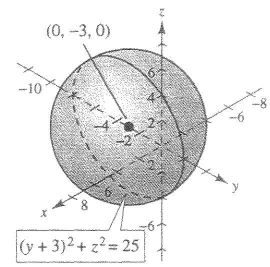
Center: $\left(\frac{1}{2}, 4, -1\right)$

Radius: 3

65. xz-trace $(y = 0)$: $(x-1)^2 + z^2 = 36$, Circle

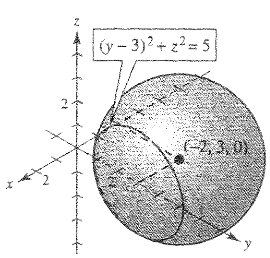


66. yz-trace $(x = 0)$: $(y+3)^2 + z^2 = 25$, Circle

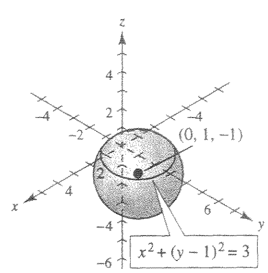


67. yz-trace $(x = 0)$: $(y-3)^2 + z^2 = 9 - 4 = 5$,

Circle

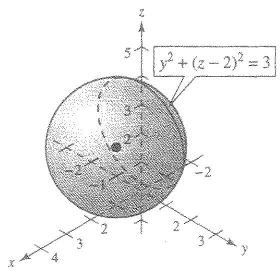


68. xy-trace $(z = 0)$: $x^2 + (y-1)^2 = 3$, Circle



69. $(x^2 - 2x + 1) + y^2 + (z^2 - 4z + 4) = -1 + 1 + 4$

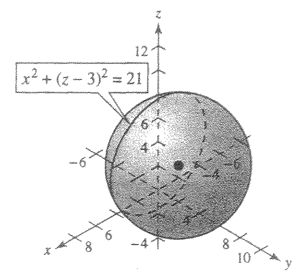
$$(x - 1)^2 + y^2 + (z - 2)^2 = 4$$



yz − trace: $x = 0$: $y^2 + (z - 2)^2 = 3$

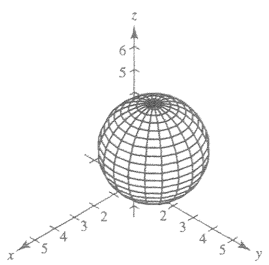
70. $x^2 + (y^2 - 4y + 4) + (z^2 - 6z + 9) = 12 + 4 + 9$

$$x^2 + (y - 2)^2 + (z - 3)^2 = 25$$



xz − trace: $y = 0$: $x^2 + (z - 3)^2 = 21$

71.

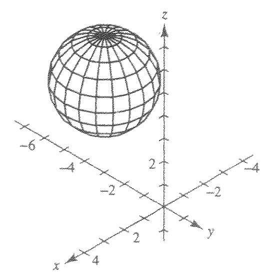


72. $x^2 + y^2 + 6y + (z^2 - 8z + 16) = -21 + 16$

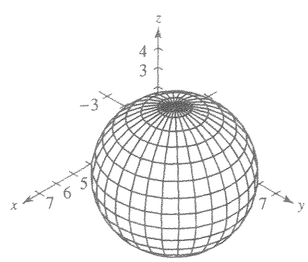
$$x^2 + y^2 + 6y + (z - 4)^2 = -5$$

$$z_1 = 4 + \sqrt{-5 - x^2 - y^2 - 6y}$$

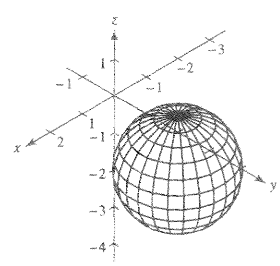
$$z_2 = 4 - \sqrt{-5 - x^2 - y^2 - 6y}$$



73.



74.



75. The length of each side is 3. Thus, $(x, y, z) = (3, 3, 3)$.

76. $x = 4, y = 4, z = 8, \quad (4, 4, 8)$

77. $d = 165 \Rightarrow r = \frac{165}{2} = 82.5$

$$x^2 + y^2 + z^2 = \left(\frac{165}{2}\right)^2$$

78. (a) $x^2 + y^2 + z^2 = 3963^2$

(b) Assume the north and south poles are on the z-axis. Lines of longitude that run north–south are traces of planes containing the z-axis. These shapes are circles of radius 3963 miles.

(c) Latitudes are traces of planes perpendicular to the z-axis. These shapes are circles.

79. False. x is the directed distance from the yz-plane to P.

80. False. The trace could be a single point, or empty.

81. In the xy-plane, the z-coordinate is 0.
In the xz-plane, the y-coordinate is 0.
In the yz-plane, the x-coordinate is 0.

82. It is a plane.

83. The trace is a circle, or a single point.

84. The trace will be a line in the xy-plane (unless the plane is the xy-plane).

85. $x_m = \dfrac{x_2 + x_1}{2} \Rightarrow x_2 = 2x_m - x_1$

Similarly for y_2 and z_2,

$(x_2, y_2, z_2) = (2x_m - x_1, 2y_m - y_1, 2z_m - z_1)$.

86. $x_2 = 2x_m - x_1 = 2(5) - 3 = 7$

$y_2 = 2y_m - y_1 = 2(8) - 0 = 16$

$z_2 = 2z_m - z_1 = 2(7) - 2 = 12$

$(7, 16, 12)$

87. $v^2 + 3v + \dfrac{9}{4} = 2 + \dfrac{9}{4}$

$\left(v + \dfrac{3}{2}\right)^2 = \dfrac{17}{4}$

$v + \dfrac{3}{2} = \pm\dfrac{\sqrt{17}}{2}$

$v = -\dfrac{3}{2} \pm \dfrac{\sqrt{17}}{2}$

88. $z^2 - 7z + \dfrac{49}{4} = 19 + \dfrac{49}{4}$

$\left(z - \dfrac{7}{2}\right)^2 = \dfrac{125}{4}$

$z - \dfrac{7}{2} = \pm\dfrac{5\sqrt{5}}{2}$

$z = \dfrac{7}{2} \pm \dfrac{5}{2}\sqrt{5}$

89. $x^2 - 5x + \dfrac{25}{4} = -5 + \dfrac{25}{4}$

$\left(x - \dfrac{5}{2}\right)^2 = \dfrac{5}{4}$

$x - \dfrac{5}{2} = \pm\dfrac{\sqrt{5}}{2}$

$x = \dfrac{5}{2} \pm \dfrac{\sqrt{5}}{2}$

90. $x^2 + 3x + \dfrac{9}{4} = 1 + \dfrac{9}{4}$

$\left(x + \dfrac{3}{2}\right)^2 = \dfrac{13}{4}$

$x + \dfrac{3}{2} = \pm\dfrac{\sqrt{13}}{2}$

$x = \dfrac{-3}{2} \pm \dfrac{\sqrt{13}}{2}$

91. $4y^2 + 4y = 9$

$y^2 + y + \dfrac{1}{4} = \dfrac{9}{4} + \dfrac{1}{4}$

$\left(y + \dfrac{1}{2}\right)^2 = \dfrac{10}{4}$

$y + \dfrac{1}{2} = \pm\dfrac{\sqrt{10}}{2}$

$y = -\dfrac{1}{2} \pm \dfrac{\sqrt{10}}{2}$

92. $x^2 + \dfrac{5}{2}x + \dfrac{25}{16} = 4 + \dfrac{25}{16}$

$\left(x + \dfrac{5}{4}\right)^2 = \dfrac{89}{16}$

$x + \dfrac{5}{4} = \pm\dfrac{\sqrt{89}}{4}$

$x = \dfrac{-5}{4} \pm \dfrac{\sqrt{89}}{4}$

93. $\mathbf{v} = 3\mathbf{i} - 3\mathbf{j}$, Quadrant IV

$\|\mathbf{v}\| = \sqrt{3^2 + (-3)^2}$

$= \sqrt{18}$

$= 3\sqrt{2}$

$\tan\theta = -\dfrac{3}{3} = -1 \Rightarrow$

$\theta = -45°$ or $315°$

94. $\mathbf{v} = \langle -1, 2\rangle$, Quadrant II

$\|\mathbf{v}\| = \sqrt{1^2 + 2^2} = \sqrt{5}$

$\tan\theta = \dfrac{2}{-1} \Rightarrow \theta \approx 116.6°$

95. $\mathbf{v} = 4\mathbf{i} + 5\mathbf{j}$, Quadrant I

$\|\mathbf{v}\| = \sqrt{16 + 25} = \sqrt{41}$

$\tan\theta = \dfrac{5}{4} \Rightarrow \theta \approx 51.34°$

96. $\mathbf{v} = \langle 10, -7\rangle$, Quadrant IV

$\|\mathbf{v}\| = \sqrt{100 + 49} = \sqrt{149}$

$\tan\theta = \dfrac{-7}{10} \Rightarrow \theta \approx 325.0°$

97. $\mathbf{u} \cdot \mathbf{v} = \langle -4, 1\rangle \cdot \langle 3, 5\rangle$

$= -4(3) + 1(5)$

$= -7$

98. $\mathbf{u} \cdot \mathbf{v} = \langle -1, 0\rangle \cdot \langle -2, -6\rangle$

$= 2 + 0$

$= 2$

99. $a_0 = 1, a_n = a_{n-1} + n^2$

$a_1 = 1 + 1^2 = 2$

$a_2 = 2 + 2^2 = 6$

$a_3 = 6 + 3^2 = 15$

$a_4 = 15 + 4^2 = 31$

	1		2		6		15		31
First differences:		1		4		9		16	
Second differences:			3		5		7		

Neither model

100. $a_0 = 0, a_n = a_{n-1} - 1$

$a_1 = 0 - 1 = -1$

$a_2 = -1 - 1 = -2$

$a_3 = -3$

$a_4 = -4$

	0		−1		−2		−3		−4
First differences:		−1		−1		−1		−1	
Second differences:			0		0		0		

Linear model

101. $a_1 = -1, a_n = a_{n-1} + 3$

$a_2 = -1 + 3 = 2$

$a_3 = 2 + 3 = 5$

$a_4 = 5 + 3 = 8$

$a_5 = 8 + 3 = 11$

	−1		2		5		8		11
First differences:		3		3		3		3	
Second differences:			0		0		0		

Linear model

102. $a_1 = 4, a_n = a_{n-1} - 2n$

$a_2 = 4 - 2(2) = 0$

$a_3 = 0 - 2(3) = -6$

$a_4 = -6 - 2(4) = -14$

$a_5 = -14 - 2(5) = -24$

	4		0		−6		−14		−24
First differences:		−4		−6		−8		−10	
Second differences:			−2		−2		−2		

Quadratic model

103. $(x+5)^2 + (y-1)^2 = 49$

104. $(x-3)^2 + (y+6)^2 = 81$

105. $(y-1)^2 = 4p(x-4),\ p = -3$

$(y-1)^2 = 4(-3)(x-4)$

$(y-1)^2 = -12(x-4)$

106. $(x-h)^2 = 4p(y-k),\ p = -5, (h, k) = (-2, 5)$

$(x+2)^2 = 4(-5)(y-5)$

$(x+2)^2 = -20(y-5)$

107. $a = 3, b = 2$, center: $(3, 3)$, horizontal major axis

$$\frac{(x-3)^2}{9} + \frac{(y-3)^2}{4} = 1$$

108. Center: $(0, 3)$

Vertical major axis length 9 $\implies a = \frac{9}{2}$

$c = 3 \implies b^2 = a^2 - c^2 = \frac{81}{4} - 9 = \frac{45}{4}$

$$\frac{(x-0)^2}{(45/4)} + \frac{(y-3)^2}{(81/4)} = 1$$

109. Center: $(6, 0)$, horizontal transverse axis

$a = 2, c = 6, b^2 = c^2 - a^2 = 36 - 4 = 32$

$$\frac{(x-6)^2}{4} - \frac{y^2}{32} = 1$$

110. Center: $(3, 5)$, vertical transverse axis

$a = 4, c = 5, b^2 = c^2 - a^2 = 25 - 16 = 9$

$$\frac{(y-5)^2}{16} - \frac{(x-3)^2}{9} = 1$$

Section 10.2 Vectors in Space

- Vectors in space $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ have many of the same properties as vectors in the plane.
- The dot product of two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ in space is $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3$.
- Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are said to be parallel if there is some scalar c such that $\mathbf{u} = c\mathbf{v}$.
- You should be able to use vectors to solve real life problems.

Vocabulary Check

1. zero

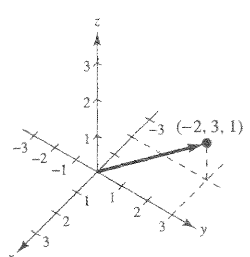
2. $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$

3. component form

4. orthogonal

5. parallel

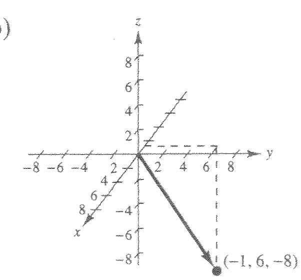
1. $\mathbf{v} = \langle 0-2, 3-0, 2-1 \rangle = \langle -2, 3, 1 \rangle$



2. (a) $\mathbf{v} = \langle 0-1, 4-(-2), -4-4 \rangle$

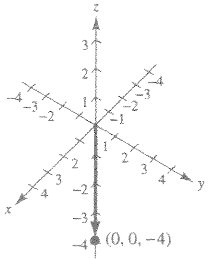
$= \langle -1, 6, -8 \rangle$

(b)



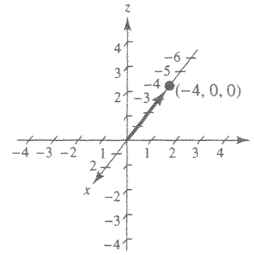
3. (a) $\mathbf{v} = \langle 1-1, 4-4, 0-4\rangle = \langle 0, 0, -4\rangle$

(b)



4. (a) $\mathbf{v} = \langle 0-4, -2-(-2), 1-1\rangle = \langle -4, 0, 0\rangle$

(b)



5. (a) $\mathbf{v} = \langle 1-(-6), -1-4, 3-(-2)\rangle$

$= \langle 7, -5, 5\rangle$

(b) $\|\mathbf{v}\| = \sqrt{7^2 + (-5)^2 + 5^2}$

$= \sqrt{49 + 25 + 25}$

$= \sqrt{99}$

$= 3\sqrt{11}$

(c) $\dfrac{\mathbf{v}}{\|\mathbf{v}\|} = \dfrac{1}{3\sqrt{11}}\langle 7, -5, 5\rangle = \dfrac{\sqrt{11}}{33}\langle 7, -5, 5\rangle$

6. (a) $\mathbf{v} = \langle 0+7, 0-3, 2-5\rangle = \langle 7, -3, -3\rangle$

(b) $\|\mathbf{v}\| = \sqrt{49 + 9 + 9} = \sqrt{67}$

(c) Unit vector:

$\dfrac{1}{\sqrt{67}}\langle 7, -3, -3\rangle = \dfrac{\sqrt{67}}{67}\langle 7, -3, -3\rangle$

7. (a) $\mathbf{v} = \langle 1-(-1), 4-2, -4-(-4)\rangle = \langle 2, 2, 0\rangle$

(b) $\|\mathbf{v}\| = \sqrt{2^2 + 2^2 + 0^2} = \sqrt{8} = 2\sqrt{2}$

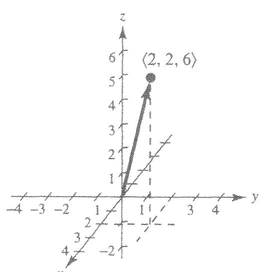
(c) Unit vector: $\dfrac{1}{2\sqrt{2}}\langle 2, 2, 0\rangle = \left\langle \dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}, 0\right\rangle$

8. (a) $\mathbf{v} = \langle 0-0, 2-(-1), 1-1\rangle = \langle 0, 3, 0\rangle$

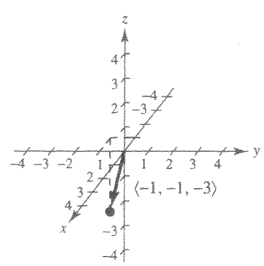
(b) $\|\mathbf{v}\| = \sqrt{0^2 + 3^2 + 0^2} = \sqrt{9} = 3$

(c) Unit vector: $\dfrac{1}{3}\langle 0, 3, 0\rangle = \langle 0, 1, 0\rangle$

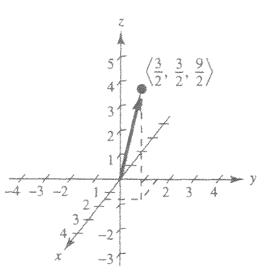
9. (a)



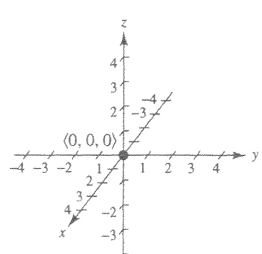
(b)



(c)

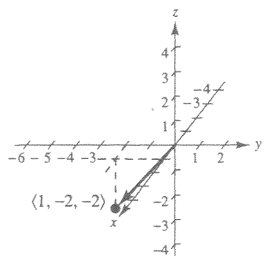


(d)

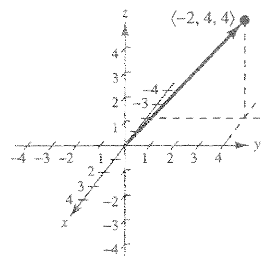


10. $\mathbf{v} = \langle -1, 2, 2 \rangle$

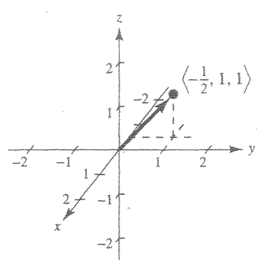
(a) $-\mathbf{v} = \langle 1, -2, -2 \rangle$



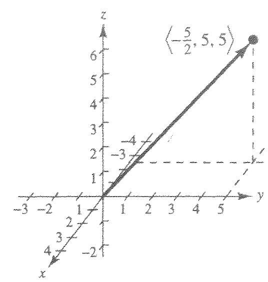
(b) $2\mathbf{v} = \langle -2, 4, 4 \rangle$



(c) $\frac{1}{2}\mathbf{v} = \left\langle -\frac{1}{2}, 1, 1 \right\rangle$

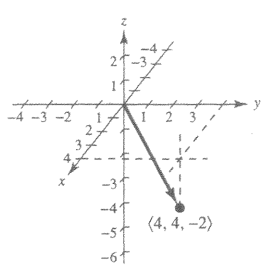


(d) $\frac{5}{2}\mathbf{v} = \left\langle \frac{-5}{2}, 5, 5 \right\rangle$

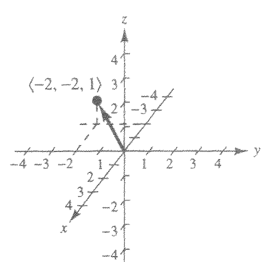


11. $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

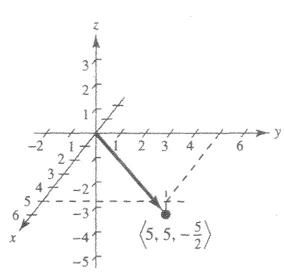
(a) $2\mathbf{v} = 4\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$



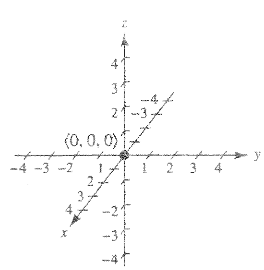
(b) $-\mathbf{v} = -2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$



(c) $\frac{5}{2}\mathbf{v} = 5\mathbf{i} + 5\mathbf{j} - \frac{5}{2}\mathbf{k}$

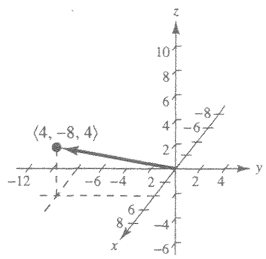


(d) $0\mathbf{v} = \mathbf{0}$

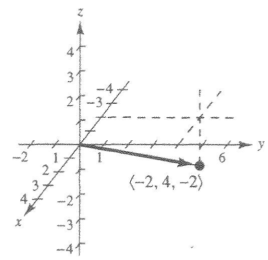


12. $\mathbf{v} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$

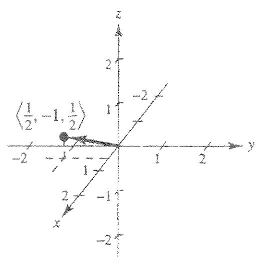
(a) $4\mathbf{v} = 4\mathbf{i} - 8\mathbf{j} + 4\mathbf{k}$



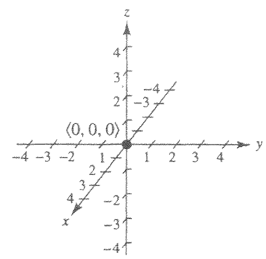
(b) $-2\mathbf{v} = -2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$



(c) $\frac{1}{2}\mathbf{v} = \frac{1}{2}\mathbf{i} - \mathbf{j} + \frac{1}{2}\mathbf{k}$



(d) $0\mathbf{v} = \mathbf{0}$



13. $\mathbf{z} = \mathbf{u} - 2\mathbf{v} = \langle -1, 3, 2\rangle - 2\langle 1, -2, -2\rangle = \langle -3, 7, 6\rangle$

14. $\mathbf{z} = 7\langle -1, 3, 2\rangle + \langle 1, -2, -2\rangle - \frac{1}{5}\langle 5, 0, -5\rangle = \langle -7, 19, 13\rangle$

15. $2\mathbf{z} - 4\mathbf{u} = \mathbf{w} \implies \mathbf{z} = \frac{1}{2}(4\mathbf{u} + \mathbf{w}) = \frac{1}{2}(4\langle -1, 3, 2\rangle + \langle 5, 0, -5\rangle) = \left\langle \frac{1}{2}, 6, \frac{3}{2}\right\rangle$

16. $\mathbf{z} = -\mathbf{u} - \mathbf{v} = -\langle -1, 3, 2\rangle - \langle 1, -2, -2\rangle = \langle 0, -1, 0\rangle$

17. $\mathbf{z} = 2\langle -1, 3, 2\rangle - 3\langle 1, -2, -2\rangle + \frac{1}{2}\langle 5, 0, -5\rangle = \left\langle -\frac{5}{2}, 12, \frac{15}{2}\right\rangle$

18. $\mathbf{z} = 3\langle 5, 0, -5\rangle - 2\langle 1, -2, -2\rangle + \langle -1, 3, 2\rangle = \langle 12, 7, -9\rangle$

19. $4\mathbf{z} = 4\langle 5, 0, -5\rangle - \langle -1, 3, 2\rangle + \langle 1, -2, -2\rangle = \langle 22, -5, -24\rangle$

$\mathbf{z} = \left\langle \frac{11}{2}, -\frac{5}{4}, -6\right\rangle$

20. $\mathbf{z} = \mathbf{w} - \mathbf{u} - 2\mathbf{v} = \langle 5, 0, -5\rangle - \langle -1, 3, 2\rangle - 2\langle 1, -2, -2\rangle = \langle 4, 1, -3\rangle$

21. $\|\mathbf{v}\| = \|\langle 7, 8, 7\rangle\|$

$= \sqrt{49 + 64 + 49} = \sqrt{162} = 9\sqrt{2}$

22. $\|\mathbf{v}\| = \sqrt{(-2)^2 + 0^2 + (-5)^2} = \sqrt{4 + 25} = \sqrt{29}$

23. $\|\mathbf{v}\| = \sqrt{1^2 + (-2)^2 + 4^2} = \sqrt{21}$

24. $\|\mathbf{v}\| = \sqrt{(-1)^2 + 0^2 + 3^2} = \sqrt{10}$

25. $\|\mathbf{v}\| = \sqrt{2^2 + (-4)^2 + 1^2} = \sqrt{21}$

26. $\|\mathbf{v}\| = \sqrt{1^2 + 3^2 + (-1)^2} = \sqrt{11}$

27. $\|\mathbf{v}\| = \sqrt{4^2 + (-3)^2 + (-7)^2}$
$= \sqrt{16 + 9 + 49} = \sqrt{74}$

28. $\|\mathbf{v}\| = \sqrt{2^2 + (-1)^2 + 6^2} = \sqrt{41}$

29. $\mathbf{v} = \langle 1 - 1, 0 - (-3), -1 - 4\rangle = \langle 0, 3, -5\rangle$
$\|\mathbf{v}\| = \sqrt{0 + 3^2 + (-5)^2} = \sqrt{34}$

30. $\mathbf{v} = \langle 1 - 0, 2 - (-1), -2 - 0\rangle = \langle 1, 3, -2\rangle$
$\|\mathbf{v}\| = \sqrt{1 + 9 + 4} = \sqrt{14}$

31. $\|\mathbf{u}\| = \sqrt{5^2 + (-12)^2} = \sqrt{169} = 13$
(a) $\frac{1}{13}(5\mathbf{i} - 12\mathbf{k})$
(b) $-\frac{1}{13}(5\mathbf{i} - 12\mathbf{k})$

32. $\|\mathbf{u}\| = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5$
(a) $\frac{1}{5}(3\mathbf{i} - 4\mathbf{k}) = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{k}$
(b) $-\frac{1}{5}(3\mathbf{i} - 4\mathbf{k}) = -\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{k}$

33. (a) $\dfrac{\mathbf{u}}{\|\mathbf{u}\|} = \dfrac{\langle 8, 3, -1\rangle}{\sqrt{74}}$
$= \dfrac{1}{\sqrt{74}}(8\mathbf{i} + 3\mathbf{j} - \mathbf{k}) = \dfrac{\sqrt{74}}{74}\langle 8, 3, -1\rangle$
(b) $-\dfrac{1}{\sqrt{74}}(8\mathbf{i} + 3\mathbf{j} - \mathbf{k}) = -\dfrac{\sqrt{74}}{74}\langle 8, 3, -1\rangle$

34. (a) $\dfrac{\mathbf{u}}{\|\mathbf{u}\|} = \dfrac{\langle -3, 5, 10\rangle}{\sqrt{134}} = \dfrac{1}{\sqrt{134}}(-3\mathbf{i} + 5\mathbf{j} + 10\mathbf{k})$
(b) $\dfrac{-1}{\sqrt{134}}(-3\mathbf{i} + 5\mathbf{j} + 10\mathbf{k})$

35. $6\mathbf{u} - 4\mathbf{v} = 6\langle -1, 3, 4\rangle - 4\langle 5, 4.5, -6\rangle = \langle -6, 18, 24\rangle + \langle -20, -18, 24\rangle = \langle -26, 0, 48\rangle$

36. $2\mathbf{u} + \frac{5}{2}\mathbf{v} = 2\langle -1, 3, 4\rangle + \frac{5}{2}\langle 5, 4.5, -6\rangle = \left\langle \frac{21}{2}, \frac{69}{4}, -7\right\rangle$

37. $\mathbf{u} + \mathbf{v} = \langle -1, 3, 4\rangle + \langle 5, 4.5, -6\rangle = \langle 4, 7.5, -2\rangle$
$\|\mathbf{u} + \mathbf{v}\| = \sqrt{4^2 + 7.5^2 + (-2)^2} = \frac{1}{2}\sqrt{305} \approx 8.73$

38. $\dfrac{\mathbf{v}}{\|\mathbf{v}\|} = \dfrac{\langle 5, 4.5, -6\rangle}{\sqrt{25 + 20.25 + 36}} = \dfrac{\langle 5, 4.5, -6\rangle}{5\sqrt{13}/2} = \left\langle \dfrac{2}{\sqrt{13}}, \dfrac{9}{5\sqrt{13}}, \dfrac{-12}{5\sqrt{13}}\right\rangle \approx \langle 0.5547, 0.4992, -0.6656\rangle$

39. $\mathbf{u} \cdot \mathbf{v} = \langle 4, 4, -1\rangle \cdot \langle 2, -5, -8\rangle$
$= 8 - 20 + 8 = -4$

40. $\mathbf{u} \cdot \mathbf{v} = 3(4) + (-1)(-10) + 6(1) = 28$

41. $\mathbf{u} \cdot \mathbf{v} = \langle 2, -5, 3\rangle \cdot \langle 9, 3, -1\rangle$
$= 18 - 15 - 3 = 0$

42. $\mathbf{u} \cdot \mathbf{v} = 0(6) + 3(-4) + (-6)(-2) = 0$

43. $\cos\theta = \dfrac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \dfrac{-8}{\sqrt{8}\sqrt{25}} \implies \theta \approx 124.45^\circ$

44. $\cos\theta = \dfrac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \dfrac{5}{\sqrt{10}\sqrt{6}} \implies \theta \approx 49.80^\circ$

45. $\cos\theta = \dfrac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \dfrac{-120}{\sqrt{1700}\sqrt{73}} \implies \theta \approx 109.92^\circ$

46. $\cos\theta = \dfrac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \dfrac{100}{\sqrt{464}\sqrt{125}} \implies \theta \approx 65.47^\circ$

47. $-\frac{3}{2}\langle 8, -4, -10\rangle = \langle -12, 6, 15\rangle \implies$ parallel

48. $\mathbf{u} \cdot \mathbf{v} = -2 - 3 - 5 = -10 \neq 0$ and
$\mathbf{u} \neq c\mathbf{v} \implies$ neither

49. $\mathbf{u} \cdot \mathbf{v} = 3 - 5 + 2 = 0 \implies$ orthogonal

50. $-8\mathbf{u} = -8\left\langle -1, \frac{1}{2}, -1 \right\rangle = \langle 8, -4, 8 \rangle = \mathbf{v} \Rightarrow$ parallel

51. $\mathbf{u} \neq c\mathbf{v}$

$\mathbf{u} \cdot \mathbf{v} = -2 - 6 \neq 0$

Neither parallel nor orthogonal

52. $\mathbf{u} \neq c\mathbf{v}$

$\mathbf{u} \cdot \mathbf{v} = 4 \neq 0$

Neither parallel nor orthogonal

53. $\mathbf{u} \cdot \mathbf{v} = -4 + 3 + 1 = 0$

Orthogonal

54. $\mathbf{u} \cdot \mathbf{v} = -2 + 3 - 1 = 0$

Orthogonal

55. $\mathbf{v} = \langle 7 - 5, 3 - 4, -1 - 1 \rangle = \langle 2, -1, -2 \rangle$

$\mathbf{u} = \langle 4 - 7, 5 - 3, 3 - (-1) \rangle = \langle -3, 2, 4 \rangle$

Since $\mathbf{u}$ and $\mathbf{v}$ are not parallel, the points are not collinear.

56. $\mathbf{v} = \langle -4 - (-2), 8 - 7, 1 - 4 \rangle = \langle -2, 1, -3 \rangle$

$\mathbf{u} = \langle 0 - (-4), 6 - 8, 7 - 1 \rangle = \langle 4, -2, 6 \rangle$

Since $\mathbf{u} = -2\mathbf{v}$, the points are collinear.

57. $\mathbf{v} = \langle -1 - 1, 2 - 3, 5 - 2 \rangle = \langle -2, -1, 3 \rangle$

$\mathbf{u} = \langle 3 - (-1), 4 - 2, -1 - 5 \rangle = \langle 4, 2, -6 \rangle$

Since $\mathbf{u} = -2\mathbf{v}$, the points are collinear.

58. $\mathbf{v} = \langle -1 - 0, 5 - 4, 6 - 4 \rangle = \langle -1, 1, 2 \rangle$

$\mathbf{u} = \langle -2 - (-1), 6 - 5, 7 - 6 \rangle = \langle -1, 1, 1 \rangle$

Since $\mathbf{u}$ and $\mathbf{v}$ are not parallel, the points are not collinear.

59. The vector $\langle 1, 2, 0 \rangle$ joining $(1, 2, 0)$ and $(0, 0, 0)$ is perpendicular to the vector $\langle -2, 1, 0 \rangle$ joining $(-2, 1, 0)$ and $(0, 0, 0)$:

$\langle 1, 2, 0 \rangle \cdot \langle -2, 1, 0 \rangle = -2 + 2 = 0$

The triangle is a right triangle.

60. Consider the vector $\langle -3, 0, 0 \rangle$ joining $(0, 0, 0)$ and $(-3, 0, 0)$ and the vector $\langle 1, 2, 3 \rangle$ joining $(1, 2, 3)$ and $(0, 0, 0)$:

$\langle -3, 0, 0 \rangle \cdot \langle 1, 2, 3 \rangle = -3 < 0$

The triangle has an obtuse angle.

Obtuse triangle

61. The three sides of the triangle are given by the vectors:

$\mathbf{u} = \langle -2, 4, -2 \rangle$

$\mathbf{v} = \langle -3, 5, -4 \rangle$

$\mathbf{w} = \langle -1, 1, -2 \rangle$

$\mathbf{u} \cdot \mathbf{v} = 34 > 0$

$\mathbf{u} \cdot \mathbf{w} = 10 > 0$

$\mathbf{v} \cdot \mathbf{w} = 16 > 0$

The triangle has three acute angles.

Acute triangle

62. Consider the vector $\langle -3, 12, 5 \rangle$ joining $(-1, 5, 8)$ and $(2, -7, 3)$, and the vector $\langle 5, 1, -9 \rangle$ joining $(4, 6, -1)$ and $(-1, 5, 8)$:

$\langle 5, 1, -9 \rangle \cdot \langle -3, 12, 5 \rangle = -48 < 0$

The triangle has an obtuse angle.

Obtuse triangle

63. $\mathbf{v} = \langle 2, -4, 7 \rangle = \langle q_1 - 1, q_2 - 5, q_3 - 0 \rangle \Rightarrow$

$$\left.\begin{aligned} 2 &= q_1 - 1 \\ -4 &= q_2 - 5 \\ 7 &= q_3 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} q_1 &= 3 \\ q_2 &= 1 \\ q_3 &= 7 \end{aligned}\right\} \Rightarrow \text{Terminal point is } (3, 1, 7).$$

64. $\langle 4, -1, -1\rangle = \langle x-6, y+4, z-3\rangle \Rightarrow (x, y, z) = (10, -5, 2)$

65. $\mathbf{v} = \left\langle 4, \frac{3}{2}, -\frac{1}{4}\right\rangle = \left\langle q_1 - 2, q_2 - 1, q_3 + \frac{3}{2}\right\rangle$

$4 = q_1 - 2 \Rightarrow q_1 = 6$

$\frac{3}{2} = q_2 - 1 \Rightarrow q_2 = \frac{5}{2}$

$-\frac{1}{4} = q_3 + \frac{3}{2} \Rightarrow q_3 = -\frac{7}{4}$

Terminal point: $\left(6, \frac{5}{2}, -\frac{7}{4}\right)$

66. $\left\langle \frac{5}{2}, -\frac{1}{2}, 4\right\rangle = \left\langle x-3, y-2, z+\frac{1}{2}\right\rangle \Rightarrow (x, y, z) = \left(\frac{11}{2}, \frac{3}{2}, \frac{7}{2}\right)$

67. $c\mathbf{u} = c\mathbf{i} + 2c\mathbf{j} + 3c\mathbf{k}$

$\|c\mathbf{u}\| = \sqrt{c^2 + 4c^2 + 9c^2} = |c|\sqrt{14} = 3 \Rightarrow$

$$c = \pm\frac{3}{\sqrt{14}} = \pm\frac{3\sqrt{14}}{14}$$

68. $\|c\,\mathbf{u}\| = |c|\,\|\mathbf{u}\| = |c|\sqrt{4+4+16} = |c|\sqrt{24} = 12$

$$\Rightarrow |c| = \frac{12}{\sqrt{24}} = \frac{6}{\sqrt{6}} = \sqrt{6} \Rightarrow c = \pm\sqrt{6}$$

69. $\mathbf{v} = \langle q_1, q_2, q_3\rangle$

Since $\mathbf{v}$ lies in the yz-plane, $q_1 = 0$. Since $\mathbf{v}$ makes an angle of 45°, $|q_2| = |q_3|$. Finally, $\|\mathbf{v}\| = 4$ implies that $q_2{}^2 + q_3{}^2 = 16$. Thus, $q_2 = q_3 = 2\sqrt{2}$ and $\mathbf{v} = \left\langle 0, 2\sqrt{2}, 2\sqrt{2}\right\rangle$, or $q_2 = 2\sqrt{2}$ and $q_3 = -2\sqrt{2}$ and $\mathbf{v} = \left\langle 0, 2\sqrt{2}, -2\sqrt{2}\right\rangle$.

70. $\mathbf{v}$ lies in xz-plane $\Rightarrow y = 0$.

$\mathbf{v} = 10\langle \sin 60°, 0, \cos 60°\rangle = \left\langle 5\sqrt{3}, 0, 5\right\rangle$, or

$\mathbf{v} = 10\langle -\sin 60°, 0, \cos 60°\rangle = \left\langle -5\sqrt{3}, 0, 5\right\rangle$

71. $\overrightarrow{PQ_1} = \left\langle 0, -24, -12\sqrt{21}\right\rangle$

$\overrightarrow{PQ_2} = \left\langle 12\sqrt{3}, 12, -12\sqrt{21}\right\rangle$

$\overrightarrow{PQ_3} = \left\langle -12\sqrt{3}, 12, -12\sqrt{21}\right\rangle$

Let $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$ be the tension on each wire. Since $\|\mathbf{F}_1\| = \|\mathbf{F}_2\| = \|\mathbf{F}_3\|$, there exists a constant c such that

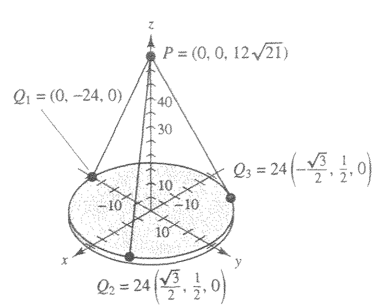
$\mathbf{F}_1 = c\left\langle 0, -24, -12\sqrt{21}\right\rangle$

$\mathbf{F}_2 = c\left\langle 12\sqrt{3}, 12, -12\sqrt{21}\right\rangle$

$\mathbf{F}_3 = c\left\langle -12\sqrt{3}, 12, -12\sqrt{21}\right\rangle$.

The total force is $-30\mathbf{k} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 \Rightarrow$ the vertical ($\mathbf{k}$) component satisfies

$$-10 = -12\sqrt{21}\,c \Rightarrow c = \frac{5}{6\sqrt{21}}.$$



$Q_1 = (0, -24, 0)$

$Q_2 = (20.8, 12, 0)$

$Q_3 = (-20.8, 12, 0)$

$P = (0, 0, 55)$

Hence,

$$\mathbf{F}_1 = \left\langle 0, \frac{-20}{\sqrt{21}}, -10\right\rangle$$

$$\mathbf{F}_2 = \left\langle \frac{10}{\sqrt{7}}, \frac{10}{\sqrt{21}}, -10\right\rangle$$

$$\mathbf{F} = \left\langle \frac{-10}{\sqrt{7}}, \frac{10}{\sqrt{21}}, -10\right\rangle$$

$\|\mathbf{F}_1\| = \|\mathbf{F}_2\| = \|\mathbf{F}_3\| \approx 10.91$ pounds.

72. $\overrightarrow{AB} = \langle 0, 70, 115\rangle$, $\mathbf{F}_1 = C_1\langle 0, 70, 115\rangle$

$\overrightarrow{AC} = \langle -60, 0, 115\rangle$, $\mathbf{F}_2 = C_2\langle -60, 0, 115\rangle$

$\overrightarrow{AD} = \langle 45, -65, 115\rangle$, $\mathbf{F}_3 = C_3\langle 45, -65, 115\rangle$

$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = \langle 0, 0, -500\rangle$. Thus

$$\begin{aligned} -60C_2 + 45C_3 &= 0 \\ 70C_1 \qquad - 65C_3 &= 0 \\ 115C_1 + 115C_2 + 115C_3 &= -500. \end{aligned}$$

Solving this system yields $C_1 = \dfrac{-104}{69}$, $C_2 = \dfrac{-28}{23}$, $C_3 = \dfrac{-112}{69}$.

Thus,

$\|\mathbf{F}_1\| \approx 202.919$ N

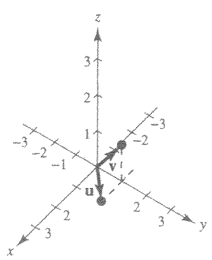
$\|\mathbf{F}_2\| \approx 157.909$ N

$\|\mathbf{F}_3\| \approx 226.521$ N.

73. True. $\cos\theta = 0 \implies \theta = 90°$

74. True

75. (a)



(b) $\mathbf{w} = a\mathbf{u} + b\mathbf{v} = a\langle 1, 1, 0\rangle + b\langle 0, 1, 1\rangle$

$\mathbf{0} = \langle a, a + b, b\rangle \implies a = b = 0$

(c) $\mathbf{w} = \langle 1, 2, 1\rangle = a\langle 1, 1, 0\rangle + b\langle 0, 1, 1\rangle$

$1 = a$

$2 = a + b$

$1 = b$

Hence, $a = b = 1$.

(d) $\mathbf{w} = \langle 1, 2, 3\rangle = a\langle 1, 1, 0\rangle + b\langle 0, 1, 1\rangle$

$1 = a$

$2 = a + b$

$3 = b$

Impossible

76. This set is a sphere.

$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = 16$

77. If $\mathbf{u} \cdot \mathbf{v} < 0$, then $\cos\theta < 0$ and the angle between $\mathbf{u}$ and $\mathbf{v}$ is obtuse, $180° > \theta > 90°$.

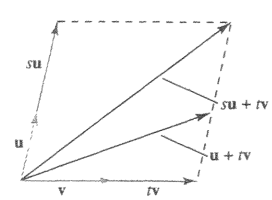
78. Let $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ and $\mathbf{u} = \langle u_1, u_2, u_3\rangle$.

Then $t\mathbf{v} = \langle tv_1, tv_2, tv_3\rangle$,

$\mathbf{u} + t\mathbf{v} = \langle u_1 + tv_1, u_2 + tv_2, u_3 + tv_3\rangle$, and

$s\mathbf{u} + t\mathbf{v} = \langle su_1 + tv_1, su_2 + tv_2, su_3 + tv_3\rangle$.

The endpoints of these three vectors are collinear, as indicated in the figure. So, the figure is a line.



79. (a) $x = t$

$y = 3t + 2$

(b) $x = t - 1$

$y = 3(t - 1) + 2 = 3t - 1$

80. (a) $x = t, y = \dfrac{2}{t}$

(b) $x = t - 1, y = \dfrac{2}{t - 1}$

81. (a) $x = t$

$y = t^2 - 8$

(b) $x = t - 1$

$y = (t - 1)^2 - 8 = t^2 - 2t - 7$

82. (a) $x = t, y = 4t^3$

(b) $x = t - 1, y = 4(t - 1)^3$

Section 10.3 The Cross Product of Two Vectors

- The cross product of two vectors $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is given by

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

- The cross product satisfies the following algebraic properties.
  (a) $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$
  (b) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$
  (c) $c(\mathbf{u} \times \mathbf{v}) = (c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v})$
  (d) $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$
  (e) $\mathbf{u} \times \mathbf{u} = \mathbf{0}$
  (f) $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$
- The following geometric properties of the cross product are valid, where θ is the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$:
  (a) $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.
  (b) $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\| \sin\theta$
  (c) $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are scalar multiples.
  (d) $\|\mathbf{u} \times \mathbf{v}\|$ is the area of the parallelogram having $\mathbf{u}$ and $\mathbf{v}$ as sides.
- The absolute value of the triple scalar product is the volume of the parallelepiped having $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ as sides.

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Vocabulary Check

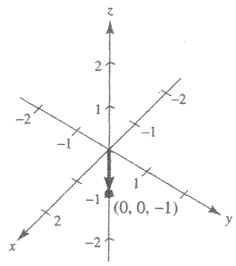
1. cross product

2. 0

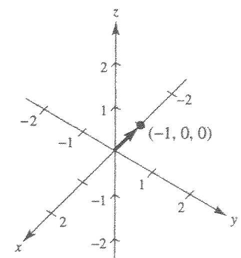
3. $\|\mathbf{u}\|\,\|\mathbf{v}\| \sin\theta$

4. triple scalar product

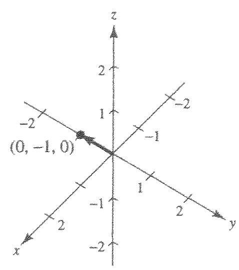
1. $\mathbf{j} \times \mathbf{i} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -\mathbf{k}$



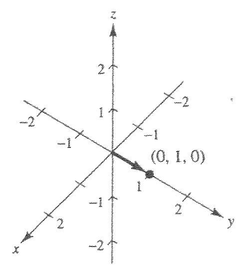
2. $\mathbf{k} \times \mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -\mathbf{i}$



3. $\mathbf{i} \times \mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -\mathbf{j}$



4. $\mathbf{k} \times \mathbf{i} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \mathbf{j}$



5. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k} = \langle 1, 1, 1 \rangle$

6. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = -\mathbf{i} - \mathbf{j} - \mathbf{k} = \langle -1, -1, -1 \rangle$

7. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 5 \\ 0 & -1 & 1 \end{vmatrix} = \langle 3, -3, -3 \rangle$

$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = \langle 3, -3, -3 \rangle \cdot \langle 3, -2, 5 \rangle = 0$

$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = \langle 3, -3, -3 \rangle \cdot \langle 0, -1, 1 \rangle = 0$

8. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 1 \\ 1 & -2 & 1 \end{vmatrix} = -\mathbf{i} - \mathbf{j} - \mathbf{k} = \langle -1, -1, -1 \rangle$

9. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -10 & 0 & 6 \\ 7 & 0 & 0 \end{vmatrix} = \langle 0, 42, 0 \rangle$

$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = \langle 0, 42, 0 \rangle \cdot \langle -10, 0, 6 \rangle = 0$

$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = \langle 0, 42, 0 \rangle \cdot \langle 7, 0, 0 \rangle = 0$

10. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -5 & 5 & 11 \\ 2 & 2 & 3 \end{vmatrix} = \langle -7, 37, -20 \rangle$

11. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 2 & 1 \\ 1 & 3 & -2 \end{vmatrix} = \langle -7, 13, 16 \rangle$

$= -7\mathbf{i} + 13\mathbf{j} + 16\mathbf{k}$

12. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & \frac{3}{2} & -\frac{5}{2} \\ \frac{1}{2} & -\frac{3}{4} & \frac{1}{4} \end{vmatrix} = \left\langle -\frac{3}{2}, -\frac{3}{2}, -\frac{3}{2} \right\rangle = -\frac{3}{2}\mathbf{i} - \frac{3}{2}\mathbf{j} - \frac{3}{2}\mathbf{k}$

13. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 4 & 3 \\ -1 & 3 & -2 \end{vmatrix} = -17\mathbf{i} + \mathbf{j} + 10\mathbf{k}$

14. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 1 \\ -2 & 1 & 2 \end{vmatrix} = -5\mathbf{i} - 8\mathbf{j} - \mathbf{k}$

15. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{2} & -\frac{2}{3} & 1 \\ -\frac{3}{4} & 1 & \frac{1}{4} \end{vmatrix} = -\frac{7}{6}\mathbf{i} - \frac{7}{8}\mathbf{j}$

16. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{2}{5} & -\frac{1}{4} & \frac{1}{2} \\ -\frac{3}{5} & 1 & \frac{1}{5} \end{vmatrix} = -\frac{11}{20}\mathbf{i} - \frac{19}{50}\mathbf{j} + \frac{1}{4}\mathbf{k}$

17. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 6 \\ -1 & 3 & 1 \end{vmatrix} = \langle -18, -6, 0 \rangle$

$= -18\mathbf{i} - 6\mathbf{j}$

18. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{2}{3} & 0 & 0 \\ 0 & \frac{1}{3} & -3 \end{vmatrix} = 2\mathbf{j} + \frac{2}{9}\mathbf{k}$

19. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & 1 \\ 0 & 1 & -2 \end{vmatrix} = \langle -1, -2, -1 \rangle$

$= -\mathbf{i} - 2\mathbf{j} - \mathbf{k}$

20. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -2 \\ 0 & -1 & 1 \end{vmatrix} = (0 - 2)\mathbf{i} - (1 - 0)\mathbf{j} + (-1 - 0)\mathbf{k} = -2\mathbf{i} - \mathbf{j} - \mathbf{k}$

21. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 4 & 3 \\ 0 & -2 & 1 \end{vmatrix} = \langle 10, -2, -4 \rangle$

22. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -2 & 6 \\ -1 & 5 & 7 \end{vmatrix} = \langle -44, -34, 18 \rangle$

23. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 4 \\ -4 & 2 & -1 \end{vmatrix} = -6\mathbf{i} - 15\mathbf{j} - 6\mathbf{k}$

24. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ -1 & 1 & -4 \end{vmatrix} = \mathbf{i} + 5\mathbf{j} + \mathbf{k}$

25. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & -5 & 1 \\ \frac{1}{2} & -\frac{3}{4} & \frac{2}{10} \end{vmatrix} = \langle -0.25, -0.7, -2 \rangle$

26. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 8 & -4 & 2 \\ \frac{1}{2} & \frac{3}{4} & -\frac{1}{4} \end{vmatrix} = -\frac{1}{2}\mathbf{i} + 3\mathbf{j} + 8\mathbf{k}$

27. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 2 & -3 & 0 \end{vmatrix} = \langle 9, 6, -7 \rangle$

$\|\mathbf{u} \times \mathbf{v}\| = \sqrt{166}$

Unit vector: $\dfrac{\sqrt{166}}{166}\langle 9, 6, -7 \rangle$

28. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ 1 & 0 & -2 \end{vmatrix} = \langle 2, 7, 1 \rangle$

$\|\mathbf{u} \times \mathbf{v}\| = \sqrt{54} = 3\sqrt{6}$

Unit vector: $\dfrac{1}{3\sqrt{6}}\langle 2, 7, 1 \rangle = \dfrac{\sqrt{6}}{18}\langle 2, 7, 1 \rangle$

29. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$

$\|\mathbf{u} \times \mathbf{v}\| = \sqrt{19}$

Unit vector $= \dfrac{\mathbf{u} \times \mathbf{v}}{\|\mathbf{u} \times \mathbf{v}\|} = \dfrac{1}{\sqrt{19}}(\mathbf{i} - 3\mathbf{j} + 3\mathbf{k})$

$= \dfrac{\sqrt{19}}{19}\langle 1, -3, 3 \rangle$

30. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 0 \\ 1 & 0 & -3 \end{vmatrix} = -6\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$

$\|\mathbf{u} \times \mathbf{v}\| = \sqrt{36 + 9 + 4} = 7$

Unit vector $= \dfrac{\mathbf{u} \times \mathbf{v}}{\|\mathbf{u} \times \mathbf{v}\|} = -\dfrac{6}{7}\mathbf{i} + \dfrac{3}{7}\mathbf{j} - \dfrac{2}{7}\mathbf{k}$

31. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 2 & -5 \\ \frac{1}{2} & -\frac{3}{4} & \frac{1}{10} \end{vmatrix} = \left\langle -\dfrac{71}{20}, -\dfrac{11}{5}, \dfrac{5}{4} \right\rangle$

Consider the parallel vector $\langle -71, -44, 25 \rangle = \mathbf{w}$.

$\|\mathbf{w}\| = \sqrt{71^2 + 44^2 + 25^2} = \sqrt{7602}$

Unit vector $= \dfrac{1}{\sqrt{7602}}\langle -71, -44, 25 \rangle$

$= \dfrac{\sqrt{7602}}{7602}\langle -71, -44, 25 \rangle$

32. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 7 & -14 & 5 \\ 14 & 28 & -15 \end{vmatrix} = 70\mathbf{i} + 175\mathbf{j} + 392\mathbf{k}$

$\|\mathbf{u} \times \mathbf{v}\| = \sqrt{70^2 + 175^2 + 392^2}$

$= \sqrt{189{,}189} = 21\sqrt{429}$

Unit vector $= \dfrac{\mathbf{u} \times \mathbf{v}}{\|\mathbf{u} \times \mathbf{v}\|} = \dfrac{1}{21\sqrt{249}}\langle 70, 175, 392 \rangle$

$= \dfrac{1}{3\sqrt{429}}\langle 10, 25, 56 \rangle$

$= \dfrac{\sqrt{429}}{1287}\langle 10, 25, 56 \rangle$

33. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = 2\mathbf{i} - 2\mathbf{j}$

$\|\mathbf{u} \times \mathbf{v}\| = 2\sqrt{2}$

Unit vector $= \dfrac{\mathbf{u} \times \mathbf{v}}{\|\mathbf{u} \times \mathbf{v}\|} = \dfrac{1}{2\sqrt{2}}(2\mathbf{i} - 2\mathbf{j})$

$= \dfrac{1}{\sqrt{2}}\mathbf{i} - \dfrac{1}{\sqrt{2}}\mathbf{j}$

$= \dfrac{\sqrt{2}}{2}\mathbf{i} - \dfrac{\sqrt{2}}{2}\mathbf{j}$

34. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 2 \\ 2 & -1 & -2 \end{vmatrix} = 6\mathbf{i} + 6\mathbf{j} + 3\mathbf{k}$

$\|\mathbf{u} \times \mathbf{v}\| = \sqrt{36 + 36 + 9} = 9$

Unit vector $= \dfrac{\mathbf{u} \times \mathbf{v}}{\|\mathbf{u} \times \mathbf{v}\|} = \dfrac{1}{9}(6\mathbf{i} + 6\mathbf{j} + 3\mathbf{k})$

$= \dfrac{2}{3}\mathbf{i} + \dfrac{2}{3}\mathbf{j} + \dfrac{1}{3}\mathbf{k}$

35. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix} = \mathbf{j}$

Area $= \|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{j}\| = 1$ square unit

36. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 2 \\ 1 & 0 & 1 \end{vmatrix} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

Area $= \|\mathbf{u} \times \mathbf{v}\| = \|2\mathbf{i} + \mathbf{j} - 2\mathbf{k}\|$

$= \sqrt{4 + 1 + 4} = 3$ square units

37. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 4 & 6 \\ 2 & -1 & 5 \end{vmatrix} = 26\mathbf{i} - 3\mathbf{j} - 11\mathbf{k}$

Area $= \|\mathbf{u} \times \mathbf{v}\| = \sqrt{26^2 + (-3)^2 + (-11)^2}$

$= \sqrt{806}$ square units

38. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 3 & 2 \\ 1 & 2 & 4 \end{vmatrix} = \langle 8, 10, -7 \rangle$

Area $= \|\mathbf{u} \times \mathbf{v}\| = \sqrt{8^2 + 10^2 + (-7)^2}$

$= \sqrt{213}$ square units

39. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & -3 \\ 0 & 2 & 3 \end{vmatrix} = \langle 12, -6, 4 \rangle$

Area $= \|\mathbf{u} \times \mathbf{v}\| = \sqrt{12^2 + (-6)^2 + 4^2}$

$= 14$ square units

40. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -3 & 2 \\ 5 & 0 & 1 \end{vmatrix} = \langle -3, 6, 15 \rangle$

Area $= \|\mathbf{u} \times \mathbf{v}\| = \sqrt{(-3)^2 + 6^2 + 15^2}$

$= \sqrt{270} = 3\sqrt{30}$ square units

41. (a) $\overrightarrow{AB} = \langle 3 - 2, 1 - (-1), 2 - 4 \rangle = \langle 1, 2, -2 \rangle$ is parallel to

$\overrightarrow{DC} = \langle 0 - (-1), 5 - 3, 6 - 8 \rangle = \langle 1, 2, -2 \rangle$.

$\overrightarrow{AD} = \langle -3, 4, 4 \rangle$ is parallel to $\overrightarrow{BC} = \langle -3, 4, 4 \rangle$.

(b) $\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -2 \\ -3 & 4 & 4 \end{vmatrix} = \langle 16, 2, 10 \rangle$

Area $= \|\overrightarrow{AB} \times \overrightarrow{AD}\|$

$= \sqrt{16^2 + 2^2 + 10^2}$

$= \sqrt{360} = 6\sqrt{10}$ square units

(c) $\overrightarrow{AB} \cdot \overrightarrow{AD} = \langle 1, 2, -2 \rangle \cdot \langle -3, 4, 4 \rangle$

$\neq 0 \implies$ not a rectangle

42. (a) $\overrightarrow{AB} = \langle 1, 2, 3 \rangle$

$\overrightarrow{CD} = \langle 1, 2, 3 \rangle$

Opposites are parallel and same length. Thus, *ABCD* form a parallelogram.

(b) $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 5 & 4 & 1 \end{vmatrix} = \langle -10, 14, -6 \rangle$

Area $= \|\overrightarrow{AB} \times \overrightarrow{AC}\| = \sqrt{(-10)^2 + 14^2 + (-6)^2} = 2\sqrt{83}$ square units

(c) $\overrightarrow{AB} \cdot \overrightarrow{AC} = 5 + 8 + 3 = 16 \neq 0 \implies$ not a rectangle

43. $\mathbf{u} = \langle 1, 2, 3 \rangle$, $\mathbf{v} = \langle -3, 0, 0 \rangle$

$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -3 & 0 & 0 \end{vmatrix} = \langle 0, -9, 6 \rangle$

Area $= \frac{1}{2}\|\mathbf{u} \times \mathbf{v}\| = \frac{1}{2}\sqrt{81 + 36} = \frac{3}{2}\sqrt{13}$

44. $\mathbf{u} = \langle 2 - 1, 0 - (-4), 2 - 3 \rangle = \langle 1, 4, -1 \rangle$

$\mathbf{v} = \langle -2 - 1, 2 - (-4), 0 - 3 \rangle = \langle -3, 6, -3 \rangle$

$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & -1 \\ -3 & 6 & -3 \end{vmatrix} = \langle -6, 6, 18 \rangle$

Area $= \frac{1}{2}\|\mathbf{u} \times \mathbf{v}\| = \frac{1}{2}\sqrt{(-6)^2 + 6^2 + 18^2}$

$= \frac{1}{2}\sqrt{396} = 3\sqrt{11}$ square units

45. $\mathbf{u} = \langle -2 - 2, -2 - 3, 0 - (-5) \rangle = \langle -4, -5, 5 \rangle$

$\mathbf{v} = \langle 3 - 2, 0 - 3, 6 - (-5) \rangle = \langle 1, -3, 11 \rangle$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & -5 & 5 \\ 1 & -3 & 11 \end{vmatrix} = \langle -40, 49, 17 \rangle$$

$\text{Area} = \frac{1}{2}\|\mathbf{u} \times \mathbf{v}\| = \frac{1}{2}\sqrt{(-40)^2 + 49^2 + 17^2}$

$= \frac{1}{2}\sqrt{4290}$ square units

46. $\mathbf{u} = \langle -2 - 2, -4 - 4, 0 - 0 \rangle = \langle -4, -8, 0 \rangle$

$\mathbf{v} = \langle 0 - 2, 0 - 4, 4 - 0 \rangle = \langle -2, -4, 4 \rangle$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & -8 & 0 \\ -2 & -4 & 4 \end{vmatrix} = \langle -32, 16, 0 \rangle$$

$\text{Area} = \frac{1}{2}\|\mathbf{u} \times \mathbf{v}\| = \frac{1}{2}\sqrt{(-32)^2 + 16^2}$

$= \frac{1}{2}\sqrt{1280} = 8\sqrt{5}$ sq. units

47. $$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 2 & 3 & 3 \\ 4 & 4 & 0 \\ 0 & 0 & 4 \end{vmatrix}$$

$= 2(16) - 3(16) + 3(0) = -16$

48. $$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 6$$

49. $$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 2 & 3 & 1 \\ 1 & -1 & 0 \\ 4 & 3 & 1 \end{vmatrix} = 2(-1) - 3(1) + 1(7) = 2$$

50. $$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 1 & 4 & -7 \\ 2 & 0 & 4 \\ 0 & -3 & 6 \end{vmatrix} = 1(0 + 12) - 4(12 - 0) - 7(-6) = 6$$

51. $$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 1 + 1 = 2$$

$\text{Volume} = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = 2$ cubic units

52. $$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 1 & 1 & 3 \\ 0 & 3 & 3 \\ 3 & 0 & 3 \end{vmatrix} = 1(9) - 1(-9) + 3(-9) = -9$$

$\text{Volume} = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = |-9| = 9$ cubic units

53. $$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 0 & 2 & 2 \\ 0 & 0 & -2 \\ 3 & 0 & 2 \end{vmatrix} = 0 - 2(6) + 2(0) = -12$$

$\text{Volume} = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = 12$ cubic units

54. $$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 1 & 2 & -1 \\ -1 & 2 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 1(2) - 2(-1 - 4) - 1(0 - 4) = 16$$

$\text{Volume} = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = 16$ cubic units

55. $\mathbf{u} = \langle 4, 0, 0\rangle,\ \mathbf{v} = \langle 0, -2, 3\rangle,\ \mathbf{w} = \langle 0, 5, 3\rangle$

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} 4 & 0 & 0 \\ 0 & -2 & 3 \\ 0 & 5 & 3 \end{vmatrix} = 4(-21) = -84$$

Volume $= |-84| = 84$ cubic units

56. $\overrightarrow{AB} = \langle 1, 1, 0\rangle, \overrightarrow{AC} = \langle 1, 0. 2\rangle, \overrightarrow{AD} = \langle 0, 1, 1\rangle$

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{vmatrix} = 1(-2) - 1(1) = -3$$

Volume $= 3$ cubic units

57. $$\mathbf{V}\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -\frac{1}{2}\cos 40^\circ & -\frac{1}{2}\sin 40^\circ \\ 0 & 0 & -p \end{vmatrix} = \left(\frac{p}{2}\cos 40^\circ\right)\mathbf{i}$$

(a) $T(p) = \|\mathbf{V}\times\mathbf{F}\| = \dfrac{p}{2}\cos 40^\circ$

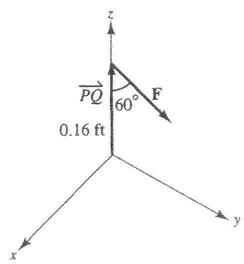
(b)

p	15	20	25	30	35	40	45
T	5.75	7.66	9.58	11.49	13.41	15.32	17.24

58. $\overrightarrow{PQ} = 0.16\mathbf{k}$

$$\overrightarrow{PQ}\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 0.16 \\ 0 & -1000\sqrt{3} & -1000 \end{vmatrix} = 160\sqrt{3}\mathbf{i}$$

$\|\overrightarrow{PQ}\times\mathbf{F}\| = 160\sqrt{3}$ ft-lb



59. True. The cross product is defined for vectors in three-dimensional space.

60. False. $\mathbf{u}\times\mathbf{v} = -(\mathbf{v}\times\mathbf{u})$

61. If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, then $\sin\theta = 1$ and hence, $\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\|\sin\theta = \|\mathbf{u}\|\,\|\mathbf{v}\|$.

62. $$\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = \mathbf{u}\times\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \mathbf{u}\times[(v_2w_3 - w_2v_3)\mathbf{i} - (v_1w_3 - w_1v_3)\mathbf{j} + (v_1w_2 - w_1v_2)\mathbf{k}]$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_2w_3 - w_2v_3 & w_1v_3 - v_1w_3 & v_1w_2 - w_1v_2 \end{vmatrix}$$

$$= [u_2(v_1w_2 - w_1v_2) - u_3(w_1v_3 - v_1w_3)]\mathbf{i}$$
$$- [u_1(v_1w_2 - w_1v_2) - u_3(v_2w_3 - w_2v_3)]\mathbf{j} + [u_1(w_1v_3 - v_1w_3) - u_2(v_2w_3 - w_2v_3)]\mathbf{k}$$

$$= [u_2w_2v_1 + u_3w_3v_1 - u_2v_2w_1 - u_3v_3w_1]\mathbf{i}$$
$$- [-u_1w_1v_2 - u_3w_3v_2 + u_1v_1w_2 + u_3v_3w_2]\mathbf{j} + [u_1w_1v_3 + u_2w_2v_3 - u_1v_1w_3 - u_2v_2w_3]\mathbf{k}$$

$$= (u_1w_1 + u_2w_2 + u_3w_3)(v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) - (u_1v_1 + u_2v_2 + u_3v_3)(w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k})$$

$$= (u_1w_1 + u_2w_2 + u_3w_3)\mathbf{v} - (u_1v_1 + u_2v_2 + u_3v_3)\mathbf{w}$$

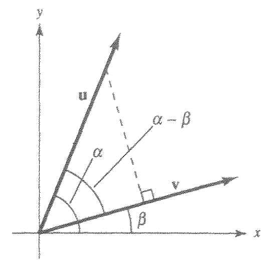
$$= (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w}$$

63. $\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = (v_2w_3 - w_2v_3)\mathbf{i} - (v_1w_3 - w_1v_3)\mathbf{j} + (v_1w_2 - v_2w_1)\mathbf{k}$

Hence,

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = u_1(v_2w_3 - w_2v_3) - u_2(v_1w_3 - w_1v_3) + u_3(v_1w_2 - v_2w_1) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

64. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \end{vmatrix} = (\cos\alpha\ \sin\beta - \sin\alpha\ \cos\beta)\mathbf{k}$



Area of triangle formed by the unit vectors $\mathbf{u}$ and $\mathbf{v}$ is

$\frac{1}{2}$(base)(height) $= \frac{1}{2}(1)\sin(\alpha - \beta)$.

The area is also given by $\frac{1}{2}\|\mathbf{u} \times \mathbf{v}\| = \frac{1}{2}|\cos\alpha \sin\beta - \sin\alpha\cos\beta|$

Notice that $\cos\alpha\sin\beta - \sin\alpha\ \cos\beta$ is negative.

Thus, $\sin(\alpha - \beta) = \sin\alpha\ \cos\beta - \cos\alpha\sin\beta$.

65. $\cos 480° = \cos 120° = -\frac{1}{2}$

66. $\tan 300° = -\sqrt{3}$

67. $\sin 690° = \sin 330° = -\frac{1}{2}$

68. $\cos 930° = \cos 210° = -\dfrac{\sqrt{3}}{2}$

69. $\sin\dfrac{19\pi}{6} = \sin\left(\dfrac{7\pi}{6}\right) = -\dfrac{1}{2}$

70. $\cos\dfrac{17\pi}{6} = \cos\dfrac{5\pi}{6} = -\dfrac{\sqrt{3}}{2}$

71. $\tan\dfrac{15\pi}{4} = \tan\dfrac{7\pi}{4} = -1$

72. $\tan\dfrac{10\pi}{3} = \tan\dfrac{4\pi}{3} = \sqrt{3}$

Section 10.4 Lines and Planes in Space

- The parametric equations of the line in space parallel to the vector $\langle a, b, c\rangle$ and passing through the point (x_1, y_2, z_3) are
  $x = x_1 + at, \quad y = y_1 + bt, \quad z = z_1 + ct.$
- The standard equation of the plane in space containing the point (x_1, y_1, z_1) and having normal vector $\langle a, b, c\rangle$ is
  $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$
- You should be able to find the angle between two planes by calculating the angle between their normal vectors.
- You should be able to sketch a plane in space.
- The distance between a point Q and a plane having normal $\mathbf{n}$ is
  $$D = \|\text{proj}_{\mathbf{n}} \overrightarrow{PQ}\| = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|}$$
  where P is a point in the plane.

Vocabulary Check

1. direction, $\dfrac{\overrightarrow{PQ}}{t}$
2. parametric equations
3. symmetric equations
4. normal
5. $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

1. $x = x_1 + at = 0 + t$

$y = y_1 + bt = 0 + 2t$

$z = z_1 + ct = 0 + 3t$

(a) Parametric equations: $x = t, y = 2t, z = 3t$

(b) Symmetric equations: $\dfrac{x}{1} = \dfrac{y}{2} = \dfrac{z}{3}$

2. $x = x_1 + at = 3 + 3t$

$y = y_1 + bt = -5 - 7t$

$z = z_1 + ct = 1 - 10t$

(a) Parametric equations:

$x = 3 + 3t, y = -5 - 7t, z = 1 - 10t$

(b) Symmetric equations: $\dfrac{x - 3}{3} = \dfrac{y + 5}{-7} = \dfrac{z - 1}{-10}$

3. $x = x_1 + at = -4 + \dfrac{1}{2}t,\ y = y_1 + bt = 1 + \dfrac{4}{3}t,\ z = z_1 + ct = 0 - t$

(a) Parametric equations: $x = -4 + \dfrac{1}{2}t, y = 1 + \dfrac{4}{3}t, z = -t$

Equivalently: $x = -4 + 3t, y = 1 + 8t, z = -6t$

(b) Symmetric equations: $\dfrac{x + 4}{3} = \dfrac{y - 1}{8} = \dfrac{z}{-6}$

4. $x = x_1 + at = 5 + 4t$

$y = y_1 + bt = 0 + 0t$

$z = z_1 + ct = 10 + 3t$

(a) Parametric equations:
$x = 5 + 4t, y = 0, z = 10 + 3t$

(b) $\dfrac{x - 5}{4} = \dfrac{z - 10}{3}, y = 0$

Not possible

5. $x = x_1 + at = 2 + 2t,$

$y = y_1 + bt = -3 - 3t,$

$z = z_1 + ct = 5 + t$

(a) Parametric equations:
$x = 2 + 2t, y = -3 - 3t, z = 5 + t$

(b) Symmetric equations: $\dfrac{x - 2}{2} = \dfrac{y + 3}{-3} = z - 5$

6. $\mathbf{v} = \langle 3, -2, 1 \rangle$

(a) $x = 1 + 3t, y = -2t, z = 1 + t$

(b) Symmetric equations: $\dfrac{x - 1}{3} = \dfrac{y}{-2} = \dfrac{z - 1}{1}$

7. $\mathbf{v} = \langle 1 - 2, 4 - 0, -3 - 2 \rangle = \langle -1, 4, -5 \rangle$

Point: $(2, 0, 2)$

(a) $x = 2 - t, y = 4t, z = 2 - 5t$

(b) $\dfrac{x - 2}{-1} = \dfrac{y}{4} = \dfrac{z - 2}{-5}$

8. $\mathbf{v} = \langle 8, 5, 12 \rangle$

Point: $(2, 3, 0)$

(a) Parametric equations:
$x = 2 + 8t, y = 3 + 5t, z = 12t$

(b) Symmetric equations: $\dfrac{x - 2}{8} = \dfrac{y - 3}{5} = \dfrac{z}{12}$

9. $\mathbf{v} = \langle 1 - (-3), -2 - 8, 16 - 15 \rangle = \langle 4, -10, 1 \rangle$

Point: $(-3, 8, 15)$

(a) $x = -3 + 4t, y = 8 - 10t, z = 15 + t$

(b) $\dfrac{x + 3}{4} = \dfrac{y - 8}{-10} = \dfrac{z - 15}{1}$

10. $\mathbf{v} = \langle 1 - 2, -5 - 3, 3 + 1\rangle = \langle -1, -8, 4\rangle$

Point: $(2, 3, -1)$

(a) $x = 2 - t, y = 3 - 8t, z = -1 + 4t$

(b) $\dfrac{x - 2}{-1} = \dfrac{y - 3}{-8} = \dfrac{z + 1}{4}$

11. $\mathbf{v} = \langle -1 - 3, 1 - 1, 5 - 2\rangle = \langle -4, 0, 3\rangle$

Point: $(3, 1, 2)$

(a) $x = 3 - 4t, y = 1, z = 2 + 3t$

(b) $\dfrac{x - 3}{-4} = \dfrac{z - 2}{3}, \; y = 1$

Not possible

12. $\mathbf{v} = \langle 2 - 2, 1 + 1, -3 - 5\rangle = \langle 0, 2, -8\rangle$

Point: $(2, -1, 5)$

(a) $x = 2, y = -1 + 2t, z = 5 - 8t$

(b) $\dfrac{y + 1}{2} = \dfrac{z - 5}{-8}, \; x = 2$

Not possible

13. $\mathbf{v} = \left\langle 1 + \dfrac{1}{2}, -\dfrac{1}{2} - 2, 0 - \dfrac{1}{2}\right\rangle = \left\langle \dfrac{3}{2}, -\dfrac{5}{2}, -\dfrac{1}{2}\right\rangle$

or $\langle 3, -5, -1\rangle$

Point: $\left(-\dfrac{1}{2}, 2, \dfrac{1}{2}\right)$

(a) $x = -\dfrac{1}{2} + 3t, y = 2 - 5t, z = \dfrac{1}{2} - t$

(b) $\dfrac{x + \frac{1}{2}}{3} = \dfrac{y - 2}{-5} = \dfrac{z - \frac{1}{2}}{-1}$

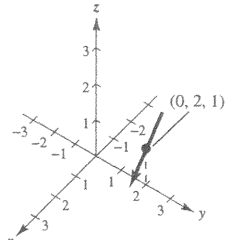
14. $\mathbf{v} = \left\langle 3 - \left(-\dfrac{3}{2}\right), -5 - \dfrac{3}{2}, -4 - 2\right\rangle = \left\langle \dfrac{9}{2}, -\dfrac{13}{2}, -6\right\rangle$, or $\langle 9, -13, -12\rangle$

Point: $(3, -5, -4)$

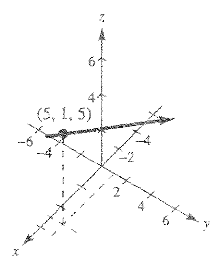
(a) Parametric equations: $x = 3 + 9t, y = -5 - 13t, z = -4 - 12t$

(b) Symmetric equations: $\dfrac{x - 3}{9} = \dfrac{y + 5}{-13} = \dfrac{z + 4}{-12}$

15.



16.



17. $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

$1(x - 2) + 0(y - 1) + 0(z - 2) = 0$

$x - 2 = 0$

18. $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

$0(x - 1) + 0(y - 0) + 1(z + 3) = 0$

$z + 3 = 0$

19. $-2(x - 5) + 1(y - 6) - 2(z - 3) = 0$

$-2x + y - 2z + 10 = 0$

20. $0(x - 0) - 3(y - 0) + 5(z - 0) = 0$

$-3y + 5z = 0$

21. $\mathbf{n} = \langle -1, -2, 1\rangle \implies$

$-1(x - 2) - 2(y - 0) + 1(z - 0) = 0$

$-x - 2y + z + 2 = 0$

22. $\mathbf{n} = \langle -1, 1, -2\rangle$

$-1(x - 0) + 1(y - 0) - 2(z - 6) = 0$

$-x + y - 2z + 12 = 0$

23. $\mathbf{u} = \langle 1-0, 2-0, 3-0\rangle = \langle 1, 2, 3\rangle$

$\mathbf{v} = \langle -2-0, 3-0, 3-0\rangle = \langle -2, 3, 3\rangle$

$$\mathbf{n} = \mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -2 & 3 & 3 \end{vmatrix} = \langle -3, -9, 7\rangle$$

$$-3(x-0) - 9(y-0) + 7(z-0) = 0$$
$$-3x - 9y + 7z = 0$$
$$3x + 9y - 7z = 0$$

24. $\mathbf{u} = \langle 2, -6, 2\rangle$, $\mathbf{v} = \langle -3, -3, 0\rangle$

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -6 & 2 \\ -3 & -3 & 0 \end{vmatrix} = \langle 6, -6, -24\rangle$$

$\mathbf{n} = \langle -1, 1, 4\rangle$

Plane: $-1(x-4) + 1(y+1) + 4(z-3) = 0$

$$-x + y + 4z - 7 = 0$$

25. $\mathbf{u} = \langle 3-2, 4-3, 2+2\rangle = \langle 1, 1, 4\rangle$

$\mathbf{v} = \langle 1-2, -1-3, 0+2\rangle = \langle -1, -4, 2\rangle$

$$\mathbf{n} = \mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 4 \\ -1 & -4 & 2 \end{vmatrix} = \langle 18, -6, -3\rangle$$

$$18(x-2) - 6(y-3) - 3(z+2) = 0$$
$$18x - 6y - 3z - 24 = 0$$
$$6x - 2y - z - 8 = 0$$

26. $\mathbf{u} = \langle 4, 0, 2\rangle$, $\mathbf{v} = \langle 1, 2, -5\rangle$

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 0 & 2 \\ 1 & 2 & -5 \end{vmatrix} = \langle -4, 22, 8\rangle$$

$\mathbf{n} = \langle -2, 11, 4\rangle$

Plane: $-2(x-1) + 11(y+1) + 4(z-2) = 0$

$$-2x + 11y + 4z + 5 = 0$$

27. $\mathbf{n} = \mathbf{j}$: $0(x-2) + 1(y-5) + 0(z-3) = 0$

$$y - 5 = 0$$

28. $\mathbf{n} = \langle 1, 0, 0\rangle$, normal to yz-plane

$$1(x-1) + 0(y-2) + 0(z-3) = 0$$
$$x - 1 = 0$$

29. $\langle 0-(-1), 2-(-2), 4-0\rangle = \langle 1, 4, 4\rangle$ and $\langle 1, 0, 0\rangle$ are parallel to the plane.

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & 4 \\ 1 & 0 & 0 \end{vmatrix} = \langle 0, 4, -4\rangle$$

$$0(x-0) + 4(y-2) - 4(z-4) = 0$$
$$4y - 4z + 8 = 0$$
$$y - z + 2 = 0$$

30. $\langle 4-1, 0-(-2), -1-4\rangle = \langle 3, 2, -5\rangle$ and $\langle 0, 1, 0\rangle$ are parallel to the plane.

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & -5 \\ 0 & 1 & 0 \end{vmatrix} = \langle 5, 0, 3\rangle$$

$$5(x-1) + 0(y+2) + 3(z-4) = 0$$
$$5x + 3z - 17 = 0$$

31. $\langle -1-2, 1-2, -1-1\rangle = \langle -3, -1, -2\rangle$ and $\langle 2, -3, 1\rangle$ are parallel to plane.

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & -1 & -2 \\ 2 & -3 & 1 \end{vmatrix} = \langle -7, -1, 11\rangle$$

$$-7(x-2) - 1(y-2) + 11(z-1) = 0$$
$$-7x - y + 11z + 5 = 0$$

32. $\langle 1-(-1), 2-(-1), 0-2\rangle = \langle 2, 3, -2\rangle$ and $\langle 2, -3, 1\rangle$ are parallel to the plane.

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -2 \\ 2 & -3 & 1 \end{vmatrix} = \langle -3, -6, -12\rangle$$

$$-3(x-1) - 6(y-2) - 12(z-0) = 0$$
$$-3x - 6y - 12z + 15 = 0$$
$$x + 2y + 4z - 5 = 0$$

33. $\mathbf{v} = \langle 0, 0, 1\rangle$ and $P = (2, 3, 4)$

$x = 2$

$y = 3$

$z = 4 + t$

34. $\mathbf{v} = \langle 0, 1, 0\rangle$ and $P = (-4, 5, 2)$

$x = -4$

$y = 5 + t$

$z = 2$

35. $\mathbf{v} = \langle 3, 2, -1\rangle$ and $P = (2, 3, 4)$

$x = 2 + 3t$

$y = 3 + 2t$

$z = 4 - t$

36. $\mathbf{v} = \langle -1, 2, 1\rangle$ and $P = (-4, 5, 2)$

$x = -4 - t$

$y = 5 + 2t$

$z = 2 + t$

37. $\mathbf{v} = \langle 2, -1, 3\rangle$ and $P = (5, -3, -4)$

$x = 5 + 2t$

$y = -3 - t$

$z = -4 + 3t$

38. $\mathbf{v} = \langle 5, -1, 0\rangle$ and $P = (-1, 4, -3)$

$x = -1 + 5t$

$y = 4 - t$

$z = -3$

39. $\mathbf{v} = \langle -1, 1, 1\rangle$ and $P = (2, 1, 2)$

$x = 2 - t$

$y = 1 + t$

$z = 2 + t$

40. $\mathbf{v} = \langle -2, 2, 0\rangle$ and $P = (-6, 0, 8)$

$x = -6 - 2t$

$y = 2t$

$z = 8$

41. $\mathbf{n}_1 = \langle 5, -3, 1\rangle, \mathbf{n}_2 = \langle 1, 4, 7\rangle$

$\mathbf{n}_1 \cdot \mathbf{n}_2 = 5 - 12 + 7 = 0$; orthogonal

42. $\mathbf{n}_1 = \langle 3, 1, -4\rangle, \mathbf{n}_2 = \langle -9, -3, 12\rangle$

$3\mathbf{n}_1 = \langle 9, 3, -12\rangle = -\mathbf{n}_2 \implies$ parallel planes

43. $\mathbf{n}_1 = \langle 2, 0, -1\rangle, \mathbf{n}_2 = \langle 4, 1, 8\rangle$

$\mathbf{n}_1 \cdot \mathbf{n}_2 = 8 - 8 = 0$; orthogonal

44. $\mathbf{n}_1 = \langle 1, -5, -1\rangle$

$\mathbf{n}_2 = \langle 5, -25, -5\rangle = 5\mathbf{n}_1 \implies$ parallel

45. (a) $\mathbf{n}_1 = \langle 3, -4, 5\rangle, \mathbf{n}_2 = \langle 1, 1, -1\rangle$; normal vectors to planes

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|} = \frac{|-6|}{\sqrt{50}\sqrt{3}} = \frac{6}{\sqrt{150}} \implies \theta \approx 60.67^\circ$$

(b) $3x - 4y + 5z = 6$ Equation 1

$x + y - z = 2$ Equation 2

(-3) times Equation 2 added to Equation 1 gives

$-7y + 8z = 0$

$$y = \frac{8}{7}z.$$

Substituting back into Equation 2, $x = 2 - y + z = 2 - \frac{8}{7}z + z = 2 - \frac{1}{7}z.$

Letting $t = z/7$, we obtain $x = 2 - t, y = 8t, z = 7t.$

46. (a) $\mathbf{n}_1 = \langle 1, -3, 1\rangle, \mathbf{n}_2 = \langle 2, 0, 5\rangle$

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|} = \frac{|7|}{\sqrt{11}\sqrt{29}} = \frac{7}{\sqrt{319}} \implies \theta \approx 66.93^\circ$$

—CONTINUED—

46. —CONTINUED—

(b) $2x + 5z + 3 = 0 \implies x = \frac{1}{2}(-5z - 3)$

Then $3y = x + z + 2 = \frac{1}{2}(-5z - 3) + z + 2 = -\frac{3}{2}z + \frac{1}{2} \implies y = -\frac{1}{2}z + \frac{1}{6}$.

Let $z = t$. Parametric equations: $x = -\frac{5}{2}t - \frac{3}{2}, y = -\frac{1}{2}t + \frac{1}{6}, z = t$

Or equivalently, let $z = 2t$ and you obtain $x = -5t - \frac{3}{2}, y = -t + \frac{1}{6}, z = 2t$.

47. (a) $\mathbf{n}_1 = \langle 1, 1, -1 \rangle, \mathbf{n}_2 = \langle 2, -5, -1 \rangle$; normal vectors to planes

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|} = \frac{|-2|}{\sqrt{3}\sqrt{30}} = \frac{2}{\sqrt{90}} \implies \theta \approx 77.83^\circ$$

(b) $x + y - z = 0$ Equation 1

$2x - 5y - z = 1$ Equation 2

(-2) times Equation 1 added to Equation 2 gives

$-7y + z = 1$

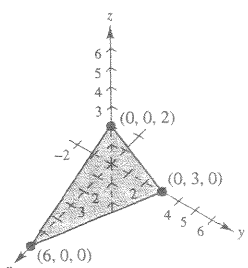
$$y = \frac{z-1}{7}.$$

Substituting back into Equation 1, $x = z - y = z - \frac{z-1}{7} = \frac{6z}{7} + \frac{1}{7} = \frac{1}{7}(6z + 1)$.

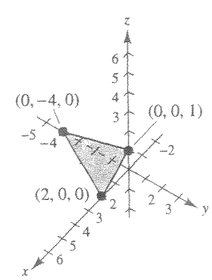
Letting $z = t$, $x = \frac{6t+1}{7}, y = \frac{t-1}{7}$. Equivalently, let $y = t, z = 7t + 1$ and $x = 6t + 1$.

48. The planes are parallel because $\mathbf{n}_1 = \langle 2, 4, -2 \rangle$ is a multiple of $\mathbf{n}_2 = \langle -3, -6, 3 \rangle$. The planes do not intersect.

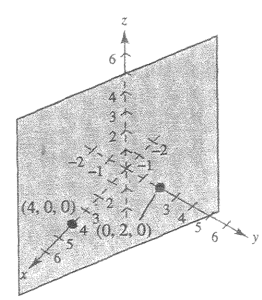
49. $x + 2y + 3z = 6$



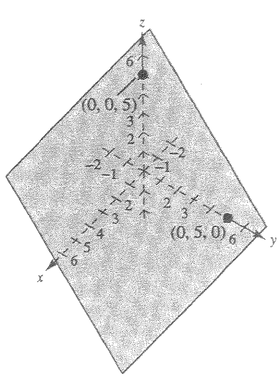
50. $2x - y + 4z = 4$



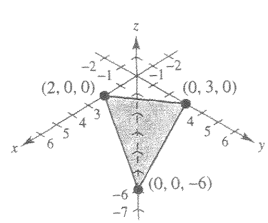
51. $x + 2y = 4$



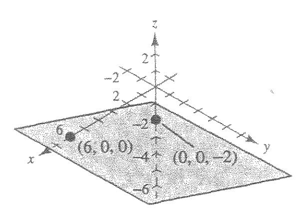
52. $y + z = 5$



53. $3x + 2y - z = 6$



54. $x - 3z = 6$



55. $D = \dfrac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|}$

$P = (1, 0, 0)$ on plane, $Q = (0, 0, 0)$,

$\mathbf{n} = \langle 8, -4, 1\rangle, \overrightarrow{PQ} = \langle -1, 0, 0\rangle$

$$D = \frac{|\langle -1, 0, 0\rangle \cdot \langle 8, -4, 1\rangle|}{\sqrt{64 + 16 + 1}} = \frac{|-8|}{\sqrt{81}} = \frac{8}{9}$$

56. $P = (4, 0, 0)$ on plane, $Q = (3, 2, 1)$, $\mathbf{n} = \langle 1, -1, 2\rangle$

$\overrightarrow{PQ} = \langle -1, 2, 1\rangle$

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{|-1|}{\sqrt{6}} = \frac{1}{\sqrt{6}} = \frac{\sqrt{6}}{6}$$

57. $D = \dfrac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|}$

$P = (2, 0, 0)$ on plane, $Q = (4, -2, -2)$,

$\mathbf{n} = \langle 2, -1, 1\rangle, \overrightarrow{PQ} = \langle 2, -2, -2\rangle$

$$D = \frac{|\langle 2, -2, -2\rangle \cdot \langle 2, -1, 1\rangle|}{\sqrt{6}} = \frac{4}{\sqrt{6}} = \frac{2\sqrt{6}}{3}$$

58. $P = (6, 0, 0)$ on plane, $Q = (-1, 2, 5)$,

$\overrightarrow{PQ} = \langle -7, 2, 5\rangle$, $\mathbf{n} = \langle 2, 3, 1\rangle$

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{|-3|}{\sqrt{14}} = \frac{3}{\sqrt{14}} = \frac{3\sqrt{14}}{14}$$

59. The normal vector to plane containing $(0, 0, 0)$, $(2, 2, 12)$ and $(10, 0, 0)$ is obtained as follows.

$\mathbf{v}_1 = \langle 2, 2, 12\rangle, \mathbf{v}_2 = \langle 10, 0, 0\rangle$

$$\mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 12 \\ 10 & 0 & 0 \end{vmatrix} = \langle 0, 120, -20\rangle$$

$\mathbf{n}_1 = \langle 0, 6, -1\rangle$

The normal vector to the plane containing $(0, 0, 0)$, $(2, 2, 12)$ and $(0, 10, 0)$ is obtained as follows.

$\mathbf{u}_1 = \langle 2, 2, 12\rangle, \mathbf{u}_2 = \langle 0, 10, 0\rangle$

$$\mathbf{u}_1 \times \mathbf{u}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 12 \\ 0 & 10 & 0 \end{vmatrix} = \langle -120, 0, 20\rangle$$

$\mathbf{n}_2 = \langle -6, 0, 1\rangle$

The angle θ between two adjacent sides is given by

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|} = \frac{|-1|}{\sqrt{37}\sqrt{37}} = \frac{1}{37} \implies \theta \approx 88.45°.$$

60. The plane containing $P(6, 0, 0)$, $S(0, 0, 0)$, $T(-1, -1, 8)$ has normal vector

$$\langle 6, 0, 0\rangle \times \langle -1, -1, 8\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 0 & 0 \\ -1 & -1 & 8 \end{vmatrix} = \langle 0, -48, -6\rangle \text{ or } \mathbf{n}_1 = \langle 0, 8, 1\rangle.$$

The plane containing $P(6, 0, 0)$, $Q(6, 6, 0)$, and $R(7, 7, 8)$ has normal vector

$$\langle 0, -6, 0\rangle \times \langle 1, 1, 8\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -6 & 0 \\ 1 & 1 & 8 \end{vmatrix} = \langle -48, 0, 6\rangle, \text{ or } \mathbf{n}_2 = \langle -8, 0, 1\rangle.$$

The angle between two adjacent sides is given by

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|} = \frac{1}{\sqrt{65}\sqrt{65}} = \frac{1}{65} \implies \theta \approx 89.12°.$$

61. False. They might be skew lines, such as:

L_1: $x = t, y = 0, z = 0$ (x-axis)

and L_2: $x = 0, y = t, z = 1$

62. True

63. The lines are parallel: $-\frac{3}{2}\langle 10, -18, 20\rangle = \langle -15, 27, -30\rangle$

64. (a) Sphere: $(x-4)^2 + (y+1)^2 + (z-1)^2 = 4$

(b) Two planes parallel to given plane. Let $Q = (x, y, z)$ be a point on one of these planes, and pick $P = (0, 0, 10)$ on the given plane. By the distance formula,

$$2 = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{|\langle x, y, z-10\rangle \cdot \langle 4, -3, 1\rangle|}{\sqrt{26}}$$

$$\pm 2\sqrt{26} = 4x - 3y + z - 10$$

$$4x - 3y + z = 10 \pm 2\sqrt{26} \quad \text{(Two planes parallel to given plane)}$$

65. $x^2 + y^2 = 10^2 = 100$

66. $\theta = \dfrac{3\pi}{4} \implies \tan\theta = -1 = \dfrac{y}{x} \implies y = -x$ (line)

67. $r = 3\cos\theta, r^2 = 3r\cos\theta, x^2 + y^2 = 3x$

68. $r = \dfrac{1}{2 - \cos\theta} \implies 2r - r\cos\theta = 1 \implies 2\sqrt{x^2+y^2} - x = 1$

$$\implies 2\sqrt{x^2+y^2} = x + 1 \implies 4(x^2+y^2) = x^2 + 2x + 1 \implies 3x^2 + 4y^2 = 2x + 1$$

69. $r^2 = 49$

$r = 7$

70. $x^2 + y^2 - 4x = 0$

$r^2 - 4r\cos\theta = 0$

$r - 4\cos\theta = 0 \implies r = 4\cos\theta$

71. $y = 5$

$r\sin\theta = 5$

$r = 5\csc\theta$

72. $2x - y + 1 = 0$

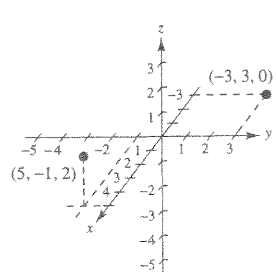
$2r\cos\theta - r\sin\theta = -1$

$r(2\cos\theta - \sin\theta) = -1$

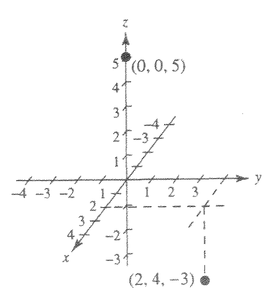
$$r = \frac{1}{\sin\theta - 2\cos\theta}$$

Review Exercises for Chapter 10

1. (a) and (b)



2.



3. $(-5, 4, 0)$

4. y-axis $\Rightarrow$ $x = z = 0$

$(0, -7, 0)$

5. $$\begin{aligned} d &= \sqrt{(5-4)^2 + (2-0)^2 + (1-7)^2} \\ &= \sqrt{1 + 4 + 36} \\ &= \sqrt{41} \end{aligned}$$

6. $$\begin{aligned} d &= \sqrt{(2-(-1))^2 + (3-(-3))^2 + (-4-0)^2} \\ &= \sqrt{9 + 36 + 16} \\ &= \sqrt{61} \end{aligned}$$

7. $$\begin{aligned} d_1 &= \sqrt{(3-0)^2 + (-2-3)^2 + (0-2)^2} = \sqrt{9+25+4} = \sqrt{38} \\ d_2 &= \sqrt{(0-0)^2 + (5-3)^2 + (-3-2)^2} = \sqrt{4+25} = \sqrt{29} \\ d_3 &= \sqrt{(0-3)^2 + (5-(-2))^2 + (-3-0)^2} = \sqrt{9+49+9} = \sqrt{67} \\ d_1{}^2 &+ d_2{}^2 = 38 + 29 = 67 = d_3{}^2 \end{aligned}$$

8. $$\begin{aligned} d_1 &= \sqrt{(4-0)^2 + (3-0)^2 + (2-4)^2} = \sqrt{16+9+4} = \sqrt{29} \\ d_2 &= \sqrt{(4-4)^2 + (5-3)^2 + (5-2)^2} = \sqrt{4+9} = \sqrt{13} \\ d_3 &= \sqrt{(4-0)^2 + (5-0)^2 + (5-4)^2} = \sqrt{16+25+1} = \sqrt{42} \\ d_1{}^2 &+ d_2{}^2 = d_3{}^2 = 42 \end{aligned}$$

9. Midpoint: $\left(\frac{-2+2}{2}, \frac{3-5}{2}, \frac{2+(-2)}{2}\right) = (0, -1, 0)$

10. Midpoint: $\left(\frac{7+1}{2}, \frac{1-1}{2}, \frac{-4+2}{2}\right) = (4, 0, -1)$

11. Midpoint: $\left(\frac{10-8}{2}, \frac{6-2}{2}, \frac{-12-6}{2}\right) = (1, 2, -9)$

12. Midpoint: $\left(\frac{-5-7}{2}, \frac{-3-9}{2}, \frac{1-5}{2}\right) = (-6, -6, -2)$

13. $(x-2)^2 + (y-3)^2 + (z-5)^2 = 1$

14. $(x-3)^2 + (y+2)^2 + (z-4)^2 = 16$

15. Radius: 6

$(x-1)^2 + (y-5)^2 + (z-2)^2 = 36$

16. Radius $= \frac{15}{2}$

$x^2 + (y-4)^2 + (z+1)^2 = \frac{225}{4}$

17. $$\begin{aligned} (x^2 - 4x + 4) + (y^2 - 6y + 9) + z^2 &= -4 + 4 + 9 \\ (x-2)^2 + (y-3)^2 + z^2 &= 9 \end{aligned}$$

Center: $(2, 3, 0)$

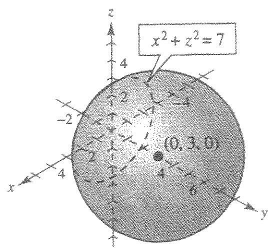
Radius: 3

18. $$\begin{aligned} (x^2 - 10x + 25) + (y^2 + 6y + 9) + (z^2 - 4z + 4) &= -34 + 25 + 9 + 4 \\ (x-5)^2 + (y+3)^2 + (z-2)^2 &= 4 \end{aligned}$$

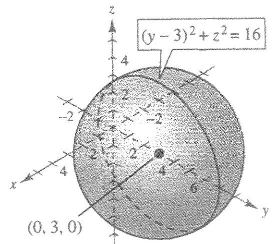
Center: $(5, -3, 2)$

Radius: 2

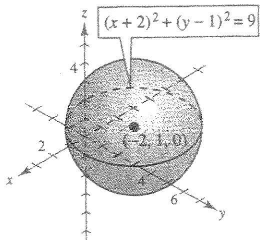
19. (a) xz-trace $(y = 0)$: $x^2 + z^2 = 7$, circle



(b) yz-trace $(x = 0)$: $(y - 3)^2 + z^2 = 16$, circle

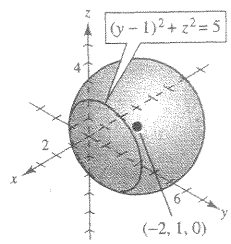


20. (a) xy-trace $(z = 0)$: $(x + 2)^2 + (y - 1)^2 = 9$, circle



(b) yz-trace $(x = 0)$: $4 + (y - 1)^2 + z^2 = 9$

$(y - 1)^2 + z^2 = 5$, circle



21. (a) $\mathbf{v} = \langle 3 - 2, 3 - (-1), 0 - 4\rangle = \langle 1, 4, -4\rangle$

(b) $\|\mathbf{v}\| = \sqrt{1^2 + 4^2 + (-4)^2} = \sqrt{33}$

(c) Unit vector: $\dfrac{\sqrt{33}}{33}\langle 1, 4, -4\rangle$

22. (a) $\mathbf{v} = \langle -3 - 2, 2 - (-1), 3 - 2\rangle = \langle -5, 3, 1\rangle$

(b) $\|\mathbf{v}\| = \sqrt{(-5)^2 + 3^2 + 1^2} = \sqrt{35}$

(c) Unit vector: $\dfrac{\sqrt{35}}{35}\langle -5, 3, 1\rangle$

23. (a) $\mathbf{v} = \langle -3 - 7, 2 - (-4), 10 - 3\rangle = \langle -10, 6, 7\rangle$

(b) $\|\mathbf{v}\| = \sqrt{(-10)^2 + 6^2 + 7^2} = \sqrt{185}$

(c) Unit vector: $\dfrac{\sqrt{185}}{185}\langle -10, 6, 7\rangle$

24. (a) $\mathbf{v} = \langle 5 - 0, -8 - 3, 6 - (-1)\rangle = \langle 5, -11, 7\rangle$

(b) $\|\mathbf{v}\| = \sqrt{5^2 + (-11)^2 + 7^2} = \sqrt{195}$

(c) Unit vector: $\dfrac{\sqrt{195}}{195}\langle 5, -11, 7\rangle$

25. $\mathbf{u} \cdot \mathbf{v} = -1(0) + 4(-6) + 3(5) = -9$

26. $\mathbf{u} \cdot \mathbf{v} = 8(2) - 4(5) + 2(2) = 0$

27. $\mathbf{u} \cdot \mathbf{v} = 2(1) - 1(0) + 1(-1) = 1$

28. $\mathbf{u} \cdot \mathbf{v} = 2(1) + 1(-3) - 2(2) = -5$

29. $\cos\theta = \dfrac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \dfrac{12 - 2 - 10}{\sqrt{42}\sqrt{17}} = 0$

$\theta \approx 90^\circ$

The vectors are orthogonal.

30. $\cos\theta = \dfrac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \dfrac{-20 - 5 - 45}{\sqrt{350}\sqrt{14}} = \dfrac{-70}{70} = -1$

$\theta \approx 180^\circ$

The vectors are parallel.

31. Since $\mathbf{u} \cdot \mathbf{v} = 0$, the angle is 90°.

32. $\cos\theta = \dfrac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \dfrac{12 + 5 - 2}{\sqrt{11}\sqrt{45}}$

$= \dfrac{15}{\sqrt{11}\sqrt{45}} \implies \theta \approx 47.61^\circ$

33. $\mathbf{u} \cdot \mathbf{v} = 7(-1) + (-2)(4) + 3(5) = 0$

Orthogonal

34. $-4\mathbf{u} = -4\langle -4, 3, -6 \rangle = \langle 16, -12, 24 \rangle = \mathbf{v}$

Parallel

35. Since $-\frac{2}{3}\langle 39, -12, 21 \rangle = \langle -26, 8, -14 \rangle$, the vectors are parallel.

36. $\mathbf{u} \cdot \mathbf{v} = \langle 8, 5, -8 \rangle \cdot \left\langle -2, 4, \frac{1}{2} \right\rangle$

$= -16 + 20 - 4 = 0$

Orthogonal

37. First two points: $\mathbf{u} = \langle -3, 4, 1 \rangle$

Last two points: $\mathbf{v} = \langle 0, -2, 6 \rangle$

Since $\mathbf{u} \neq c\mathbf{v}$, the points are not collinear.

38. First two points: $\langle -1, 5, 4 \rangle$

Last two points: $\langle 2, -10, -8 \rangle$

Since, $\langle 2, -10, -8 \rangle = -2\langle -1, 5, 4 \rangle$, the three points are collinear.

39. First two points: $\langle 4, -2, -10 \rangle$

First and third points: $\langle 2, -1, -5 \rangle$

Since $\langle 4, -2, -10 \rangle = 2\langle 2, -1, -5 \rangle$, the three points are collinear.

40. First two points: $\langle 3, -1, -2 \rangle$

Last two points: $\langle 3, 11, -2 \rangle$

Since $\langle 3, -1, -2 \rangle \neq c\langle 3, 11, -2 \rangle$, the three points are not collinear.

41. Let **a**, **b**, and **c** be the three force vectors determined by $A(0, 10, 10)$, $B(-4, -6, 10)$, and $C(4, -6, 10)$.

$$\mathbf{a} = \|\mathbf{a}\| \frac{\langle 0, 10, 10 \rangle}{10\sqrt{2}} = \|\mathbf{a}\| \left\langle 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

$$\mathbf{b} = \|\mathbf{b}\| \frac{\langle -4, -6, 10 \rangle}{\sqrt{152}} = \|\mathbf{b}\| \left\langle \frac{-2}{\sqrt{38}}, \frac{-3}{\sqrt{38}}, \frac{5}{\sqrt{38}} \right\rangle$$

$$\mathbf{c} = \|\mathbf{c}\| \frac{\langle 4, -6, 10 \rangle}{\sqrt{152}} = \|\mathbf{c}\| \left\langle \frac{2}{\sqrt{38}}, \frac{-3}{\sqrt{38}}, \frac{5}{\sqrt{38}} \right\rangle$$

Must have $\mathbf{a} + \mathbf{b} + \mathbf{c} = 300\mathbf{k}$. Thus,

$$\frac{-2}{\sqrt{38}}\|\mathbf{b}\| + \frac{2}{\sqrt{38}}\|\mathbf{c}\| = 0$$

$$\frac{1}{\sqrt{2}}\|\mathbf{a}\| - \frac{3}{\sqrt{38}}\|\mathbf{b}\| - \frac{3}{\sqrt{38}}\|\mathbf{c}\| = 0$$

$$\frac{1}{\sqrt{2}}\|\mathbf{a}\| + \frac{5}{\sqrt{38}}\|\mathbf{b}\| + \frac{5}{\sqrt{38}}\|\mathbf{c}\| = 300.$$

From the first equation, $\|\mathbf{b}\| = \|\mathbf{c}\|$. From the second equation, $\frac{1}{\sqrt{2}}\|\mathbf{a}\| = \frac{6}{\sqrt{38}}\|\mathbf{b}\|$.

From the third equation, $\frac{1}{\sqrt{2}}\|\mathbf{a}\| = 300 - \frac{10}{\sqrt{38}}\|\mathbf{b}\|$. Thus,

$$\frac{6}{\sqrt{38}}\|\mathbf{b}\| = 300 - \frac{10}{\sqrt{38}}\|\mathbf{b}\| \implies \frac{16}{\sqrt{38}}\|\mathbf{b}\| = 300 \text{ and } \|\mathbf{b}\| = \|\mathbf{c}\| = \frac{75\sqrt{38}}{4} \approx 115.58.$$

Finally, $\|\mathbf{a}\| = \sqrt{2}\left(\frac{6}{\sqrt{38}}\right)\left(\frac{75\sqrt{38}}{4}\right) = \frac{225\sqrt{2}}{2} \approx 159.10.$

42. Let **a**, **b**, **c** be the three force vectors determined by $A(0, 10, 10)$, $B(-4, -6, 10)$, and $C(4, -6, 10)$.

$$\mathbf{a} = \|\mathbf{a}\|\frac{\langle 0, 10, 10\rangle}{10\sqrt{2}} = \|\mathbf{a}\|\left\langle 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle$$

$$\mathbf{b} = \|\mathbf{b}\|\frac{\langle -4, -6, 10\rangle}{\sqrt{152}} = \|\mathbf{b}\|\left\langle \frac{-2}{\sqrt{38}}, \frac{-3}{\sqrt{38}}, \frac{5}{\sqrt{38}}\right\rangle$$

$$\mathbf{c} = \|\mathbf{c}\|\frac{\langle 4, -6, 10\rangle}{\sqrt{152}} = \|\mathbf{c}\|\left\langle \frac{2}{\sqrt{38}}, \frac{-3}{\sqrt{38}}, \frac{5}{\sqrt{38}}\right\rangle$$

We must have $\mathbf{a} + \mathbf{b} + \mathbf{c} = 200\mathbf{k}$. Thus,

$$\frac{-2}{\sqrt{38}}\|\mathbf{b}\| + \frac{2}{\sqrt{38}}\|\mathbf{c}\| = 0$$

$$\frac{1}{\sqrt{2}}\|\mathbf{a}\| - \frac{3}{\sqrt{38}}\|\mathbf{b}\| - \frac{3}{\sqrt{38}}\|\mathbf{c}\| = 0$$

$$\frac{1}{\sqrt{2}}\|\mathbf{a}\| + \frac{5}{\sqrt{38}}\|\mathbf{b}\| + \frac{5}{\sqrt{38}}\|\mathbf{c}\| = 200$$

Solving this system, $\|\mathbf{a}\| \approx 106.1$, $\|\mathbf{b}\| = \|\mathbf{c}\| = 77.1$. Thus, the tensions are 106.1, 77.1 and 77.1 pounds.

43. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 8 & 2 \\ 1 & 1 & -1 \end{vmatrix} = \langle -10, 0, -10\rangle$

44. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 10 & 15 & 5 \\ 5 & -3 & 0 \end{vmatrix} = \langle 15, 25, -105\rangle$

45. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 2 & -5 \\ 10 & -15 & 2 \end{vmatrix} = \langle -71, -44, 25\rangle$

$\|\mathbf{u} \times \mathbf{v}\| = \sqrt{7602}$

Unit vector: $\dfrac{1}{\sqrt{7602}}\langle -71, -44, 25\rangle$

46. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 4 \\ 1 & 0 & 12 \end{vmatrix} = 4\mathbf{j} \implies$ unit vector: $\mathbf{j} = \langle 0, 1, 0\rangle$

47. First two points: $\langle 3, 2, 3\rangle$

Last two points: $\langle 3, 2, 3\rangle$

First and third points: $\langle -2, 2, 0\rangle$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 3 \\ -2 & 2 & 0 \end{vmatrix} = \langle -6, -6, 10\rangle$$

$$\begin{aligned} \text{Area} &= \|\langle -6, -6, 10\rangle\| \\ &= \sqrt{36 + 36 + 100} \\ &= \sqrt{172} \\ &= 2\sqrt{43} \text{ square units} \end{aligned}$$

48. $\mathbf{u} = \langle 1, 0, 1\rangle$, $\mathbf{v} = \langle 1, 0, 1\rangle$,

Opposite sides parallel and equal length

Adjacent sides: $\mathbf{u} = \langle 1, 0, 1\rangle$, $\mathbf{w} = \langle 0, 2, 0\rangle$

$$\mathbf{u} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{vmatrix} = \langle -2, 0, 2\rangle$$

Area $= \|\mathbf{u} \times \mathbf{w}\| = \sqrt{4 + 4} = 2\sqrt{2}$ square units

49. The parallelogram is determined by the three vectors with initial point (0, 0, 0).

$\mathbf{u} = \langle 3, 0, 0\rangle, \mathbf{v} = \langle 2, 0, 5\rangle, \mathbf{w} = \langle 0, 5, 1\rangle$

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} 3 & 0 & 0 \\ 2 & 0 & 5 \\ 0 & 5 & 1 \end{vmatrix} = -75$$

Volume $= |-75| = 75$ cubic units

50. $\mathbf{u} = \langle 2, 0. 0\rangle, \mathbf{v} = \langle 0, 4, 0\rangle, \mathbf{w} = \langle 0, 0, 6\rangle$

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{vmatrix} = 48$$

Volume $= |\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})| = 48$ cubic units

51. $\mathbf{v} = \langle 9 - 3, 11 - 0, 6 - 2\rangle = \langle 6, 11, 4\rangle$

Point: (3, 0, 2)

(a) $x = 3 + 6t,\ y = 11t,\ z = 2 + 4t$

(b) $\dfrac{x-3}{6} = \dfrac{y}{11} = \dfrac{z-2}{4}$

52. $\mathbf{v} = \langle 9, 6, 2\rangle$

Point: $(-1, 4, 3)$

(a) $x = -1 + 9t,\ y = 4 + 6t,\ z = 3 + 2t$

(b) $\dfrac{x+1}{9} = \dfrac{y-4}{6} = \dfrac{z-3}{2}$

53. $\mathbf{v} = \langle 3 + 1, 6 - 3, -1 - 5\rangle = \langle 4, 3, -6\rangle$, point: $(-1, 3, 5)$

(a) Parametric equations: $x = -1 + 4t, y = 3 + 3t, z = 5 - 6t$

(b) Symmetric equations: $\dfrac{x+1}{4} = \dfrac{y-3}{3} = \dfrac{z-5}{-6}$

54. (a) $\mathbf{v} = \langle 5, 20, -3\rangle$

$x = 5t,\ y = -10 + 20t,\ z = 3 - 3t$

(b) $\dfrac{x}{5} = \dfrac{y+10}{20} = \dfrac{z-3}{-3}$

55. Use $2\mathbf{v} = \langle -4, 5, 2\rangle$, point: (0, 0, 0).

(a) Parametric equations: $x = -4t, y = 5t, z = 2t$

(b) Symmetric equations: $\dfrac{x}{-4} = \dfrac{y}{5} = \dfrac{z}{2}$

56. (a) $\mathbf{v} = \langle 1, 1, 1\rangle$

$x = 3 + t, y = 2 + t, z = 1 + t$

(b) $\dfrac{x-3}{1} = \dfrac{y-2}{1} = \dfrac{z-1}{1}$ or

$x - 3 = y - 2 = z - 1$

57. $\mathbf{u} = \langle 5, 0, 2\rangle, \mathbf{v} = \langle 2, 3, 8\rangle$

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 0 & 2 \\ 2 & 3 & 8 \end{vmatrix} = \langle -6, -36, 15\rangle$$

$\mathbf{n} = \langle 2, 12, -5\rangle$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$
$$2(x - 0) + 12(y - 0) - 5(z - 0) = 0$$
$$2x + 12y - 5z = 0$$

58. $\mathbf{u} = \langle 5, -5, -2\rangle,\ \mathbf{v} = \langle 3, 5, 2\rangle$

$$\mathbf{n} = \mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & -5 & -2 \\ 3 & 5 & 2 \end{vmatrix} = \langle 0, -16, 40\rangle$$

Plane: $0(x + 1) - 16(y - 3) + 40(z - 4) = 0$

$$-2(y - 3) + 5(z - 4) = 0$$
$$-2y + 5z - 14 = 0$$

59. $\mathbf{n} = \mathbf{k}$, normal vector

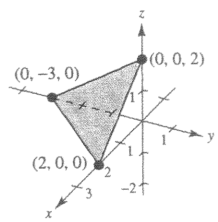
Plane: $0(x - 5) + 0(y - 3) + 1(z - 2) = 0$

$$z - 2 = 0$$

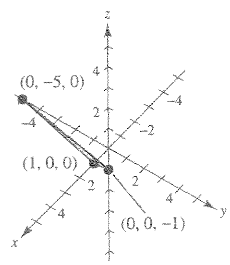
60. $\mathbf{n} = \langle -1, 1, -2 \rangle$, point: $(0, 0, 6)$

$$-1(x - 0) + 1(y - 0) - 2(z - 6) = 0$$
$$-x + y - 2z + 12 = 0$$
$$x - y + 2z - 12 = 0$$

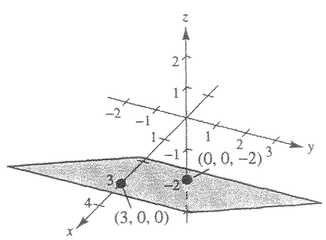
61. $3x - 2y + 3z = 6$



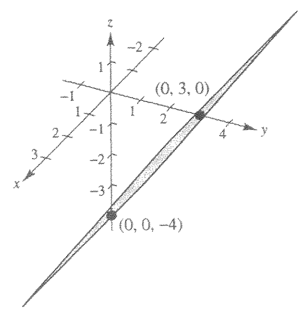
62. $5x - y - 5z = 5$



63. $2x - 3z = 6$



64. $4y - 3z = 12$



65. $\mathbf{n} = \langle 2, -20, 6 \rangle$, $P = (0, 0, 1)$ in plane, $Q = (2, 3, 10)$, $\overrightarrow{PQ} = \langle 2, 3, 9 \rangle$

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{|-2|}{\sqrt{440}} = \frac{1}{\sqrt{110}} = \frac{\sqrt{110}}{110} \approx 0.0953$$

66. $D = \dfrac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|}$

$Q = (1, 2, 3)$, $P = (2, 0, 0)$ in plane, $\overrightarrow{PQ} = \langle -1, 2, 3 \rangle$, $\mathbf{n} = \langle 2, -1, 1 \rangle$

$$D = \frac{|\langle -1, 2, 3 \rangle \cdot \langle 2, -1, 1 \rangle|}{\sqrt{6}} = \frac{1}{\sqrt{6}} = \frac{\sqrt{6}}{6}$$

67. $\mathbf{n} = \langle 1, -10, 3 \rangle$, $P = (2, 0, 0)$ in plane, $Q = (0, 0, 0)$, $\overrightarrow{PQ} = \langle -2, 0, 0 \rangle$

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{|-2|}{\sqrt{1 + 100 + 9}} = \frac{2}{\sqrt{110}} = \frac{2\sqrt{110}}{110} = \frac{\sqrt{110}}{55} \approx 0.191$$

68. $D = \dfrac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|}$

$Q = (0, 0, 0)$, $P = (0, 0, 12)$ in plane, $\overrightarrow{PQ} = \langle 0, 0, -12 \rangle$, $\mathbf{n} = \langle 2, 3, 1 \rangle$

$$D = \frac{|\langle 0, 0, -12 \rangle \cdot \langle 2, 3, 1 \rangle|}{\sqrt{14}} = \frac{12}{\sqrt{14}} = \frac{6\sqrt{14}}{7}$$

69. False. $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$

70. True. See page 761.

71. $\mathbf{u} \cdot \mathbf{u} = \langle 3, -2, 1\rangle \cdot \langle 3, -2, 1\rangle$

$$= 9 + 4 + 1$$
$$= 14$$
$$= \|\mathbf{u}\|^2$$

72. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 1 \\ 2 & -4 & -3 \end{vmatrix} = \langle 10, 11, -8\rangle$

$$\mathbf{v} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & -3 \\ 3 & -2 & 1 \end{vmatrix} = \langle -10, -11, 8\rangle$$

Thus, $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$.

73. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \langle 3, -2, 1\rangle \cdot \langle 1, -2, -1\rangle = 6$

$\mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w} = 11 + (-5) = 6$

74. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \langle 1, -2, -1\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 1 \\ 1 & -2 & -1 \end{vmatrix} = \langle 4, 4, -4\rangle$

$\mathbf{u} \times \mathbf{v} = \langle 10, 11, -8\rangle$ (Exercise 72)

$$\mathbf{u} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 1 \\ -1 & 2 & 2 \end{vmatrix} = \langle -6, -7, 4\rangle$$

$$(\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w}) = \langle 10, 11, -8\rangle + \langle -6, -7, 4\rangle = \langle 4, 4, -4\rangle$$
$$= \mathbf{u} \times (\mathbf{v} + \mathbf{w})$$

75. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$

76. See table on page 759.

77. The magnitude will increase by a factor of 4.

78. Form vectors for two sides and complete their cross product.

Chapter 10 Practice Test

1. Find the lengths of the sides of the triangle with vertices $(0, 0, 0)$, $(1, 2, -4)$, and $(0, -2, -1)$. Show that the triangle is a right triangle.

2. Find the standard form of the equation of a sphere having center $(0, 4, 1)$ and radius 5.

3. Find the center and radius of the sphere $x^2 + y^2 + z^2 + 2x - 4z - 11 = 0$.

4. Find the vector $\mathbf{u} - 3\mathbf{v}$ given $\mathbf{u} = \langle 1, 0, -1 \rangle$ and $\mathbf{v} = \langle 4, 3, -6 \rangle$.

5. Find the length of $\frac{1}{2}\mathbf{v}$ if $\mathbf{v} = \langle 2, 4, -6 \rangle$.

6. Find the dot product of $\mathbf{u} = \langle 2, 1, -3 \rangle$ and $\mathbf{v} = \langle 1, 1, -2 \rangle$.

7. Determine whether $\mathbf{u} = \langle 1, 1, -1 \rangle$ and $\mathbf{v} = \langle -3, -3, 3 \rangle$ are orthogonal, parallel, or neither.

8. Find the cross product of $\mathbf{u} = \langle -1, 0, 2 \rangle$ and $\mathbf{v} = \langle 1, -1, 3 \rangle$. What is $\mathbf{v} \times \mathbf{u}$?

9. Use the triple scalar product to find the volume of the parallelepiped having adjacent edges $\mathbf{u} = \langle 1, 1, 1 \rangle$, $\mathbf{v} = \langle 0, -1, 1 \rangle$, and $\mathbf{w} = \langle 1, 0, 4 \rangle$.

10. Find a set of parametric equations for the line through the points $(0, -3, 3)$ and $(2, -3, 4)$.

11. Find an equation of the plane passing through $(1, 2, 3)$ and perpendicular to the vector $\mathbf{n} = \langle 1, -1, 0 \rangle$.

12. Find an equation of the plane passing through the three points $A = (0, 0, 0)$, $B = (1, 1, 1)$, and $C = (1, 2, 3)$.

13. Determine whether the planes $x + y - z = 12$ and $3x - 4y - z = 9$ are parallel, orthogonal, or neither.

14. Find the distance between the point $(1, 1, 1)$ and the plane $x + 2y + z = 6$.

CHAPTER 11
Limits and an Introduction to Calculus